Interface motion for the stochastic Allen–Cahn and Cahn–Hilliard equation

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Interface motion for the stochastic
Allen–Cahn and Cahn–Hilliard equation

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Abstract

In this thesis, we consider the stochastic Cahn–Hilliard–Cook and (mass conserving) Allen–Cahn equation in the physically relevant space dimensions. Both of these equations serve as phenomenological model for the phase separation and subsequent coarsening of a two-component mixture. In our studies, we focus on the almost final stage of the evolution, when after an initial spinodal decomposition or nucleation the mixture is well-separated, and the dynamics is given by the motion of an interface on a metastable slow manifold. In the one-dimensional setting, the slow manifold is parametrized by the zeros of a profile having a finite number of transitions from one pure phase into the other. In higher space dimensions for very late stages of separation, the transition between phases occurs in a small neighborhood of an almost spherical interface. Here, the metastable manifold consists of translations of a droplet state with a fixed size.

We derive the effective equation on the slow manifold via an orthogonal projection for a relatively small noise and small atomistic interaction length. Thus, the underlying infinite-dimensional system can be described to very high accuracy by a finite-dimensional stochastic ordinary differential equation. We will see that the thermal fluctuations dominate the dynamics. This is quite different to the deterministic case, where at this stage the evolution is exponentially slow in the atomistic interaction length.

We analyze the stochastic stability and show that solutions stay close to the slow manifold for a very long time with high probability. Crucial for the stability analysis are spectral estimates of the linearization around the energetically favorable states.

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CHAPTER 1

Introduction

1.1 The Cahn–Hilliard and Allen–Cahn equation

In this thesis, we study a specific class of reaction-diffusion equations (models A and B in the theory of dynamics of critical phenomena, cf. [HH77]), which serve as phenomenological models for phase separation and subsequent coarsening of a two-component mixture. The current state of the mixture can be described by an order parameter $u(t,x)$ depending on space and time and taking its values in $[-1,1]$. The values $u = \pm 1$ correspond to the pure phases of the two components, while a value $u \in (-1,1)$ stands for mixtures of the two phases. Typical examples are the phase transition in an ice-water system at zero temperature (Allen–Cahn) and the phase separation of binary metal alloys (Cahn–Hilliard). In both scenarios, the sample prefers to be in the pure phases ± 1. In order to describe such physical systems, it is natural to take the potential energy

$$\int F(u(x))\,\mathrm{d}x,$$

where $F(\cdot)$ denotes a symmetric double-well potential with the global minima attained at ± 1. A typical example is the quartic potential $F(u) = \frac{1}{4}(u^2-1)^2$ (see Figure 1.1), but most properties are independent of the particular choice of F.

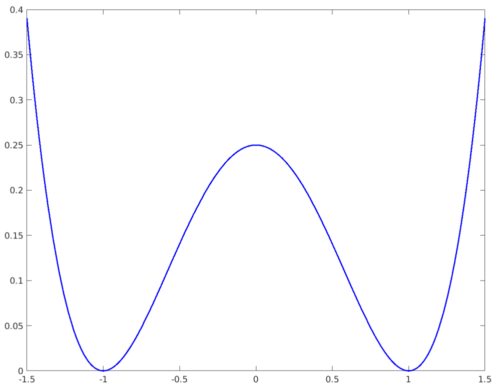

Figure 1.1.: The potential $F(u) = \frac{1}{4}(u^2-1)^2$

This description, however, is purely local, and to overcome the possibility of rough transitions between pure phases, one needs to introduce a surface term

$$\int \frac{\varepsilon^2}{2}|\nabla u(x)|^2\,\mathrm{d}x.$$

The parameter $\varepsilon > 0$ is a small atomistic interaction length that describes the typical width of a transition between two different phases. Hence, our model is depicted by the energy landscape given by the Ginzburg–Landau–Wilson free energy functional

$$J_\varepsilon(u) = \int \left(\frac{\varepsilon^2}{2} |\nabla u(x)|^2 + F(u(x)) \right) \mathrm{d}x.$$

In order to derive an equation for the phase field parameter u, it is assumed that the underlying system relaxes rapidly towards configurations that are energetically favorable for the potential J_ε, and the dynamics is governed by a gradient flow structure, i.e.,

$$\frac{d}{dt} u = -\frac{\partial J_\varepsilon(u)}{\partial u} = \varepsilon^2 \Delta u - F'(u).$$

This corresponds to the L^2-gradient flow of the energy functional $J_\varepsilon(u)$.

The two-component mixture described by this model is contained in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. Here, we allow for the physically relevant space dimensions $d = 1$ (referred to as one-dimensional case) and $d \in \{2, 3\}$ (the higher-dimensional cases). It is natural to assume a no-flux condition, that is, at no time can mass leave or enter the domain. This leads to the Allen–Cahn equation subjected to Neumann boundary conditions

$$\begin{cases} \partial_t u(t,x) = \varepsilon^2 \Delta u(t,x) - F'(u(t,x)), & x \in \Omega,\ t > 0 \\ \partial_\eta u(t,x) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{AC}$$

This equation was first introduced by Samuel M. Allen and John W. Cahn [AC79]. One key feature of the Allen–Cahn equation is that the total mass in the system is not preserved, as one has by Green's identity

$$\frac{d}{dt} \int_\Omega u(t,x) \, \mathrm{d}x = -\int_\Omega F'(u(t,x)) \, \mathrm{d}x.$$

To enforce mass conservation, one could add a correction and obtain the following non-local version of the Allen–Cahn equation

$$\begin{cases} \partial_t u(t,x) = \varepsilon^2 \Delta u(t,x) - F'(u(t,x)) + \dfrac{1}{|\Omega|} \displaystyle\int_\Omega F'(u(t,x)) \, \mathrm{d}x, & x \in \Omega,\ t > 0 \\ \partial_\eta u(t,x) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{mAC}$$

Here, $|\Omega|$ denotes the area of the bounded domain Ω and the added mean value of the function $F'(u)$ takes care of the mass constraint. Another way to guarantee mass conservation is to choose a different topology for which one considers the gradient flow of the Ginzburg–Landau energy J_ε. In fact, choosing the space L_0^2, the subspace of L^2 consisting of functions with mean zero, leads to (mAC). If one takes the space H_0^{-1}, the subspace of the Sobolev space H^{-1} with zero average, equipped with the inner product

$$\langle u, v \rangle_{H_0^{-1}} = \langle (-\Delta)^{-1/2} u, (-\Delta)^{-1/2} v \rangle_{L^2},$$

we arrive at the Cahn–Hilliard equation, which was postulated earlier in the 1950s by J. Cahn and J. Hilliard [CH58, Cah59]

$$\begin{cases} \partial_t u(t,x) = -\Delta \left(\varepsilon^2 \Delta u(t,x) - F'(u(t,x)) \right), & x \in \Omega,\ t > 0 \\ \partial_\eta u(t,x) = \partial_\eta \Delta u(t,x) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{CH}$$

As mentioned, one striking difference between the Allen–Cahn equation and the Cahn–Hilliard equation is that (CH) preserves the total mass $\int_\Omega u(t,x) \, \mathrm{d}x$ in the system, while this is not true for (AC). A mathematical difference lies in the fact that the Cahn–Hilliard equation is a fourth-order equation and thus does not allow for comparison principles, which are a useful tool in the study of the Allen–Cahn equation. In this work, we do not rely on maximum principles, and hence, the general results work for both equations.

1.2 The concept of slow manifolds and heuristics

The theory of slow manifolds is an important tool in the study of dynamical systems. It gives a practical method to reduce the degrees of freedom in a model and often results in considerable simplifications. For example, in this thesis, we start with an infinite-dimensional stochastic partial differential equation (SPDE) and end up with a finite-dimensional stochastic ordinary differential equation (SDE) describing the motion of interfaces on a metastable manifold. In some models of interest, there is a separation of time scales between some quantities that relax very quickly to an essentially static value, while others change more slowly and can be sensitive to perturbations. The term „slow manifold“ describes the space in which these slower quantities vary after a possible fast initial transient has died out. This concept is similar to the theory of center manifolds, but slow manifolds only provide an approximation.

Let us give a non-rigorous description on constructing a finite-dimensional slow manifold for the Cahn–Hilliard and Allen–Cahn equation. Due to the gradient flow structure of these equations, the preferable states of the dynamics are given by the minimizers of the energy functional J_ε. First, we take a look at the one-dimensional case on the whole line $\mathbb{R}$ with ε scaled out, i.e., in this case, the energy landscape is described by the functional

$$J(u) \;=\; \int_{\mathbb{R}} \left(\frac{1}{2}u'(x)^2 + F(u(x))\right) \mathrm{d}x.$$

Obviously, the pure phases $u = \pm 1$ are global minimizers, but—due to mass conservation for (mAC) and (CH)—a non-homogeneous material never reaches this perfect configuration. Thus, we search for minimizers under the constraint that there exists at least one transition from -1 to $+1$. For this purpose, we demand that u is increasing and satisfies $\lim_{x\to\pm\infty} u(x) = \pm 1$. Using these boundary conditions and the monotonicity, one can then write

$$\begin{aligned} J(u) \;&=\; \int_{\mathbb{R}} \left(\frac{1}{2}u'(x)^2 + F(u(x))\right) \mathrm{d}x \\ &=\; \frac{1}{2}\int_{\mathbb{R}} \left(u'(x) - \sqrt{2F(u(x))}\right)^2 + \sqrt{2F(u(x))}u'(x)\,\mathrm{d}x \;\geq\; \int_{-1}^{1} \sqrt{2F(u)}\,\mathrm{d}u \;=:\; c_0. \end{aligned}$$

In the case of the classical potential $F(u) = \frac{1}{4}(u^2-1)^2$, we obtain $c_0 = \frac{2\sqrt{2}}{3}$. This energy level is attained if, and only if, u solves the first-order equation

$$u'(x) - \sqrt{2F(u(x))} \;=\; 0 \quad \forall x \in \mathbb{R}, \qquad \lim_{x\to\pm\infty} u(x) \;=\; \pm 1. \tag{1.1}$$

Since equation (1.1) and the energy J are invariant under translations, we can also assume that the profile u is centered at $u(0) = 0$. If the potential F is sufficiently smooth, there exists a unique solution $u(x)$, $x \in \mathbb{R}$, expressed implicitly by

$$x \;=\; \int_0^{u(x)} \frac{1}{\sqrt{2F(t)}}\,\mathrm{d}t.$$

For the quartic double-well potential, we obtain the solution $u(x) = \tanh(x/\sqrt{2})$. Due to the translation invariance of (1.1), we can define the one-parameter family of solutions $\{u^\xi\}_{\xi\in\mathbb{R}}$, where $u^\xi(x) := u(x-\xi)$ is a translate of the normalized solution. By our construction, the function u^ξ is an energetically favorable profile for the energy functional J that jumps from -1 to $+1$ in a small neighborhood around the zero ξ. One key feature of the profiles u^ξ (and their derivatives) is that they converge exponentially fast to ± 1 (and 0, respectively), and thus, almost all the energy is concentrated near ξ. Moreover, due to the symmetry of the potential F, we observe that $-u^\xi$ solves equation (1.1), but the transition goes from $+1$ to -1.

Under the same constraint of having exactly one phase transition from -1 to $+1$, the minimizers of the functional J_ε in the ε-dependent case are given by a rescaled version of the profiles u^ξ, i.e.,

$$u^\xi_\varepsilon(x) \,=\, u\left(\frac{x-\xi}{\varepsilon}\right).$$

With the exponential decay of u^ξ, we also see that the phase transition occurs in a vicinity of ξ of width ε, and thus, the typical length of an interface is of order ε. In order to generate profiles with a fixed number N of transitions from -1 to $+1$, and vice versa, that occur at some zeros $\xi_1, \dots, \xi_N$, one can essentially sum up these energy minimizers with alternating sign (see Figure 1.2). If the zeros are well separated, we expect these configurations to be almost stationary as they solve the PDE up to exponentially small terms.

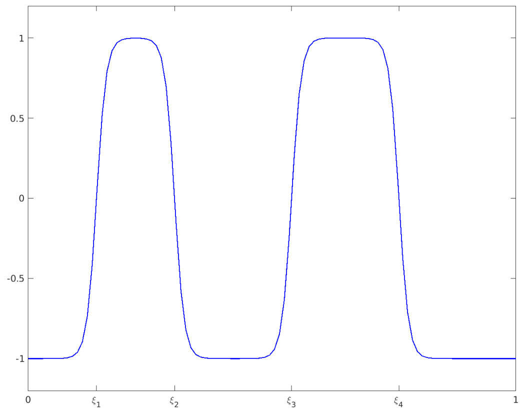

Figure 1.2.: A typical multi-kink profile with an approx. energy of $4 \cdot c_0$

In the higher dimensional cases, we can think of the domain Ω being split into subdomains Γ^+_ε and Γ^-_ε, where an energetically optimal configuration is close to the pure phases $u = +1$ or $u = -1$ with boundaries ε-localized about an interface $\Gamma_\varepsilon(t)$ between $\Gamma^\pm_\varepsilon$. For the Cahn–Hilliard equation, as ε tends to zero, the front $\Gamma_\varepsilon(t)$ moves at least locally up to time-scales of order 1 according to the geometric evolution law (cf. [Peg89, Sto96, ABC94])

$$v \,=\, b\left[\frac{d\mu}{d\eta}\right]_{\Gamma(t)}, \tag{MS}$$

where

$$\begin{aligned} \Delta\mu \,&=\, 0, \qquad && x \in \Omega \setminus \Gamma(t), \\ \frac{\partial\mu}{d\eta} \,&=\, 0, && x \in \partial\Omega, \\ \mu \,&=\, \varepsilon a K, && x \in \Gamma(t). \end{aligned}$$

Here, a and b are constants, K denotes the mean curvature of $\Gamma(t)$ at x, $[\frac{d\mu}{d\eta}]$ is the jump of the normal derivative $\frac{d\mu}{d\eta}$ across $\Gamma(t)$, and v is the normal component of the velocity of $\Gamma(t)$. Equation (MS) is referred to as Mullins–Sekerka problem. For corresponding results on the Allen–Cahn equation, we refer to [ESS92, DMS95] and the references therein. We observe that a sphere, or more generally a surface consisting of a finite number of non-overlapping spheres contained in Ω, is an equilibrium to the Mullins–Sekerka problem. This suggests that one should investigate bounded radial solutions to the stationary problem

$$\Delta u - F'(u) \,=\, 0, \quad x \in \mathbb{R}^d. \tag{1.2}$$

In fact, if $U(x) = U^*(|x|)$ is such a function, then the shifted radial component $U_\rho(s) = U^*(s+\rho)$ satisfies

$$U_\rho'' + \frac{d-1}{\rho+s}\,U_\rho - F'(U_\rho) \;=\; 0.$$

Therefore, as $\rho \to \infty$, we can expect that U_ρ tends to the one-dimensional heteroclinic given by (1.1). Moreover, away from the interface, we expect that U_ρ is close to one of the roots of F'. With a perturbation argument based on this observation, Nicholas D. Alikakos and Giorgio Fusco [AF98] proved the existence of radial solutions to (1.2). For the rescaled problem, this leads to a droplet like state $u_\varepsilon^{\xi,\rho}$ that jumps from somewhere near -1 to near 1 in a thin layer of width ε around the sphere of radius ρ and center ξ.

So far, we have seen on a heuristic level that energetically favorable configurations can be described by a finite-dimensional shape variable—the positions $\{\xi_i\}_{i=1,\dots,N}$ of the kinks in the one-dimensional setting, centers and radii $(\xi_i, \rho_i)_{i=1,\dots,N}$ of droplets in the higher dimensional cases. It is fruitful to interpret these states as finite-dimensional (smooth) manifold. For a set $\mathcal{O} \subset \mathbb{R}^N$ of admissible parameters, we denote the corresponding profile by u^ξ, i.e., we write

$$\mathcal{M} \;=\; \left\{ u^\xi \,:\, \xi \in \mathcal{O} \right\}.$$

By the gradient flow structure of the Allen–Cahn and Cahn–Hilliard equation, solutions will be drawn rapidly to a small neighborhood of the manifold $\mathcal{M}$, and then the configuration remains almost static. In the deterministic case, the kinks or droplets move exponentially slow in the atomistic interaction length ε until two of them come too close and annihilate in a fast motion (also known as Ostwald ripening). During this stage of the evolution, the total area, where transitions between the pure phases occur, decreases, and thus the domain, where the solution is constant, increases—thereby leading to a decay of energy. This phase of the dynamics is referred to as coarsening.

In the final stage, there will be only one kink or one droplet left. Due to mass conservation, the position of the last remaining kink or the size of the last droplet is fixed. Thus the one-dimensional kink does not move. As almost all the energy is stored in the interfacial region, the droplet moves slowly towards the boundary of the domain, where a semicircular shape can be obtained and thus a shortening of the perimeter.

Throughout this thesis, we will focus on a fixed number of transition layers in the one-dimensional case. Therefore, we study the dynamics locally in time, i.e., as long as the transition layers are well separated. After a breakdown, the number of transition layers has decreased (by two if they annihilate and by one if they hit the boundary), and one could restart the analysis on a lower-dimensional slow manifold. In higher space dimensions, we concentrate on the motion of a single droplet inside the domain and sufficiently far away from the boundary. We do not study the „fast“ annihilation in this thesis.

1.3 The stochastic equations

In this thesis, we study the later stage of the evolution—after spinodal decomposition or nucleation—when the two-component mixture is already well-separated, and domains of pure phases have formed. To capture a more interesting behavior for the interface motion and motivated by thermal fluctuations in the material, we introduce an additional noise term and study its influence on the dynamics.

As described above, in the deterministic case, one observes a rapid phase separation followed by a very slow dynamics on the slow manifold $\mathcal{M}$. In the one-dimensional case, for instance, the (in ε) exponentially slow motion of the kinks was established both for the Allen–Cahn equation by Carr and Pego [CP89, CP90] as well as for the Cahn–Hilliard equation by Bates and Xun [BX94, BX95]. In the stochastic setting, however, the strong decay of energy will still lead to a fast phase separation, but there is almost no influence of this energy on the slow manifold. Hence, the thermal fluctuations—although tiny—do have an impact on the dynamics. The kinks or droplets will move randomly and annihilate once they come too close. We will see that the interface positions essentially behave like on $\mathcal{M}$ projected Brownian motions, which are coupled through the mass constraint. In this sense, one expects that the additional noise term significantly accelerates the coarsening procedure.

The stochastic forcing is given by the derivative of a $\mathcal{Q}$-Wiener process. Throughout our analysis, we will assume that the covariance operator $\mathcal{Q}$ is trace-class. The motivation of this choice lies in the fact that the proofs of our main stability results rely heavily on sufficient smoothness of the solutions and Itô formula. Therefore, we need the Wiener process to be sufficiently smooth in space, too. We expect the results to remain valid for singular noise, although the region of stochastic stability might decrease. We give an approach towards treating rougher noise in Section 2.4.

1.4 Bibliographical notes

The deterministic Cahn–Hilliard equation was proposed by J. Cahn and J. Hilliard [CH58, Cah59] as a simple model for the phase separation of a binary alloy at a fixed temperature. It was extended first by H. Cook [Coo70] in the 1970s to incorporate thermal fluctuations in the form of additive white noise. See also [Lan71]. Since then, there have been many developments, and we give a brief overview of the literature. We refer to [CH58, Cah61, EZ86, Fif91, BF93] and the references therein for a more physical description, the derivation, and further discussions.

The existence and uniqueness of solutions to the stochastic problem are well-understood. It was first studied by Da Prato and Debussche in [DPD96], where the nonlinearity is given by a polynomial of odd degree with positive leading term and the problem is posed on rectangular domains. Here, the stochastic forcing is given by a space-time white noise. In [CW01], the author proved the existence of solutions and its density. For a trace-class Wiener process, the existence was analyzed in [EM91].

Our analysis focuses on the later stages of the evolution when the binary alloy is already well-separated and domains of pure phases have formed. For an analysis of earlier stages, see, for example, [Bat90, BF93, BMPW05, BGW10, BSW16]. In [BMPW05], for instance, the authors showed that for a solution starting at a homogeneous state, the probability of staying near a certain finite-dimensional space of patterns is high as long the solution stays within a certain distance of the homogeneous state.

Bates and Xun [BX94, BX95] offered a detailed analysis of the slow evolution of patterns for the one-dimensional Cahn–Hilliard equation. Building on the construction of a slow manifold due to Carr and Pego [CP89], the authors proved the existence of metastable patterns and analyzed the equations governing their motion. They studied the dynamics of an equilibrium having finitely many transitions layers and showed that the kinks move exponentially slow in the atomistic interaction length ε. In the stochastic case, the motion of kinks was studied in [ABK12]. Relying on the same deterministic slow manifold as Bates and Xun, the authors proved that with high probability solutions stay in a small tube around this manifold consisting of muli-kink profiles. Opposed to the exponentially slow motion of the kinks in the deterministic case, the stochastic terms dominate the dynamics.

In the higher-dimensional cases, the interface is expected to move like a Hele–Shaw or Mullins–Sekerka problem (see equation (MS)), where circular-shaped droplets are stable stationary solutions for the dynamics. This was first suggested by formal analysis of Pego [Peg89] and later supported rigorously in [ABC94, Sto96] in the sharp interface limit. In [ABK18], formal derivation suggested a stochastic Hele–Shaw problem in the limit $\varepsilon \to 0$ for a noise strength of order ε. There it was also shown that for small noise the dynamics is well approximated by a deterministic Hele–Shaw problem. See also [BYZ19] for singular noise. The dynamics of the interface in the sharp interface limit was also studied in [YZ19], but without obtaining an equation on the interface. Only in the case of radial symmetric interfaces, one obtains the full Hele–Shaw problem.

In [AF98, AFK04], the motion of a single spherical droplet or bubble for the deterministic Cahn–Hilliard equation inside a smooth domain was analyzed, and it was shown that the droplet moves (in ε) exponentially slow towards the closest point at the boundary. Via energy methods and a careful analysis of the spectrum, in [ABF98], the authors established slow motion for models described by the energy landscape J_ε. For a detailed analysis of the spectrum of the linearized Cahn–Hilliard operator, see [AF94]. Otto and Reznikoff [OR07] also presented a general framework for slow motion in systems given as a gradient flow.

The Allen–Cahn equation was first introduced by S. Allen and J. Cahn in [AC79]. As well as the Cahn–Hilliard equation, it serves as a phenomenological model to describe phase separation and essentially behaves in the same way, but without mass conservation.

The dynamics of the one-dimensional equation is well-understood. J. Carr and R. Pego [CP89, CP90] provided a detailed analysis of the slow evolution of patterns of the singularly perturbed Ginzburg-Landau equation. They proved the existence and persistence of metastable patterns and analyzed the equations governing their exponentially slow motion. These metastable states have been characterized in terms of the global unstable manifolds of equilibria. The idea of the metastable manifold of multi-kink configurations due to Carr and Pego [CP89] led to many further investigations. For a complete picture of the dynamics, including annihilation, we refer to the nice work of X. Chen [Che04].

Heavily based on maximum principles, the first rigorous works on the stochastic problem can be found in [Fun95, BDMP95]. Here, the sharp interface limit for a single-kink configuration is studied. See also [BB98]. In [Sha00], the author proved for a sufficiently small noise strength the convergence of solutions to the one-dimensional stochastic Allen–Cahn equation to the noise-free problem. In recent years, the evolution of a multi-kink profile was analyzed; however, only the invariant measure was considered [Web10, OWW14]. In [Web14], a martingale representation for the interfaces was given, and the effect of annihilation was studied in more detail.

In the space dimensions two and three, the Allen–Cahn equation has also been treated in the literature. As we mentioned earlier, in the higher-dimensional cases the interfaces form a $(d-1)$-dimensional surface. For the deterministic equation, the motion of the fronts according to a mean curvature flow in the sharp interface limit $\varepsilon \to 0$ was established in [Che92]. This was extended by Funaki [Fun99] to the stochastic equation for a noise that is constant in space and smoothened in time. The nonlocal version of the Allen–Cahn equation, i.e., the lack of mass conservation is taken care of by a nonlocal term, was first studied in [RS92]. Here, the authors analyzed the final stage of the evolution, where a single droplet moves to the boundary of the domain. They established a stable set of solutions corresponding to small semicircular droplets intersecting the boundary and moving towards a point of locally maximal curvature. Further works in this regard are [ACF00, BJ14]. In the stochastic case, the motion of a single boundary droplet was studied in [ABBK15].

1.5 Organization of the thesis

In this work, we analyze the motion of interfaces for the stochastic Cahn–Hilliard and Allen–Cahn equation in the relevant space dimensions $d = 1$ and $d = 2, 3$. The two- and three-dimensional problem is referred to as higher-dimensional cases and the analysis can be carried out almost analogously in these higher-dimensional cases. We focus on the later stages of the evolution when the dynamics is in first approximation given by the motion on a finite-dimensional metastable slow manifold. So far in the stochastic case, only the motion of multi-kink configurations for the one-dimensional Cahn–Hilliard equation has been carried out in [ABK12], and we extend the picture in this work to the stochastic Allen–Cahn equation in the space dimensions $d = 1, 2$ and 3, as well as the stochastic Cahn–Hilliard equation in the higher-dimensional cases. Building on preliminary works on the deterministic equations for the construction of a slow manifold, we analyze the stochastic ODE governing the motion of interfaces and show stability of the deterministic slow manifold for long times under small stochastic perturbations.

In Chapter 2, we establish the mathematical framework of this thesis. For a general infinite-dimensional stochastic system and some finite-dimensional slow manifold $\mathcal{M}$, we first compute the effective dynamics on $\mathcal{M}$. Here, to make the computation feasible, we assume that an Itô diffusion gives the motion on the slow manifold. Via orthogonal projection, we define the so-called Fermi coordinates in a small neighborhood of $\mathcal{M}$ and use them to derive the effective dynamics. In the second part, we analyze the stochastic stability of the slow manifold. Based on a method introduced in [ABK12, ABBK15], we present a general guideline to achieve stochastic stability. Crucial for our analysis are spectral properties of linearized operators. We need that eigenfunctions not tangential to the manifold have negative eigenvalues uniformly bounded away from zero.

The methods are applied to the stochastic Cahn–Hilliard equation in higher space dimensions in Chapter 3. Here, we rely on the existence of a deterministic slow manifold that consists of translations of a single-droplet state, which was constructed in [AF98]. We show that sufficiently close to the manifold, the motion of the droplet's center is approximately given by the projection of the Wiener process onto the tangent space of the slow manifold. As the dominating terms for the dynamics are not small in ε but in the radius of the bubble, we expect that the motion is influenced by the mass. Stochastic stability is derived after an extensive study of the linearization of the Cahn–Hilliard operator around a droplet state. Here, it is quite useful that the H^{-1}-bounds correspond to the L^2-theory of the linearized Allen–Cahn operator. In more detail, we prove that a weighted H^1-distance of a solution to the slow manifold stays small with high probability up to time scales that are polynomial in ε^{-1}. We conclude the analysis of the Cahn–Hilliard equation by extending the stability result to general nonlinearities that have at most polynomial growth at infinity.

To complete the picture in higher space dimensions, we study the stochastic mass conserving Allen–Cahn equation afterwards in Chapter 4. In our analysis, we can rely on the results from Chapter 3 on the Cahn–Hilliard equation. In fact, we consider the same slow manifold of droplet states. We state the exact stochastic equation for the droplet's motion and analyze it thereafter in terms of ε. Compared to the exponentially slow motion in the deterministic case, it comes as no surprise that the additional noise dominates the dynamics and the motion of the droplet behaves like a projected Wiener process. Moreover, we first prove stochastic stability in L^2 and then extend the stability result to H^1, which we need to control the nonlinearity in the stochastic ODE governing the motion of the droplet's center.

Finally, Chapter 5 is devoted to the one-dimensional stochastic Allen–Cahn equation. We treat the classical Allen–Cahn equation, as well as the nonlocal version, which preserves the total mass. To construct a slow manifold of multi-kink configurations having $N+1$ transitions from $-1 \leftrightarrow +1$ (cf. Figure 1.2), we start with the profiles u_ε^ξ given by rescaled versions of the heteroclinic (1.1). The key feature of these profiles is that they decay exponentially to ± 1. Also, all the derivatives converge exponentially fast to zero. With this observation, we can essentially define a multi-kink configuration by summing up such profiles with an alternating sign. Here, our construction of the slow manifold differs slightly from the construction in the deterministic case due to Carr and Pego [CP89, CP90]. In their work, the alternating profiles were carefully glued together via a cut-off function. With that, the authors gained better control of the exponentially small error, which is crucial since exponentially small terms dominate the dynamics. In our stochastic case, however, the dynamics is dominated by the noise, and hence, we do not need to take care of these exponentially small terms.

Due to mass conservation, the dimension of the slow manifold is reduced by one in the nonlocal case. This phenomenon can also be observed on the level of the stochastic ODE governing the motion of the kinks. While for the classical Allen–Cahn equation the kinks behave like independent Brownian motions until they come too close and annihilate, the motion for the mass conserving case is—as one would expect—coupled through the mass constraint.

With the method introduced in Chapter 2, we prove stability in L^2. In order to control the nonlinear terms in the stochastic ODE, we extend the stability result to L^4. The advantage of working only in Lebesgue spaces is that we do not need to assume additional spatial regularity of the Wiener process. This is quite different to the higher-dimensional cases. Ultimately, we treat general nonlinearities given by polynomials of odd degree $2p-1$ with positive leading term. Here, we extend the stability result to L^{2p}.

CHAPTER 2

General setting & Metatheorems

In this chapter, we consider the mathematical framework of this thesis and develop toolboxes, which will be used throughout this work. We assume that solutions to some (infinite-dimensional) stochastic PDE are well approximated by some ansatz functions u^h, which are collected in a finite-dimensional slow manifold $\mathcal{M}$ (Definition 2.1) parametrized by some shape variable $h \in \mathcal{O} \subset \mathbb{R}^N$. Under a suitable coordinate frame (Fermi coordinates, Definition 2.2), we derive the effective dynamics on $\mathcal{M}$ in Section 2.2, cf. Theorem 2.6. In Section 2.3, we are concerned with establishing stochastic stability, i.e., our objective is to show that solutions stay close to the slow manifold for very long times under small stochastic perturbations. The typical time scale for stochastic stability should be much longer than the one we expect the shape variable h to move. Motivated by [ABBK15, ABK12], we use Itô formula to estimate the differential of the squared residual error $\|u - u^h\|^2$. Crucial for the analysis are bounds on the linearization at any ansatz function $u^h \in \mathcal{M}$ orthogonal to the tangent space of $\mathcal{M}$ and good control of the nonlinear terms (Metatheorems 2–4). The main result on stochastic stability is presented in Theorem 2.14. Since our method is decisively based on the application of Itô formula, we have to assume that solutions to the stochastic PDE are sufficiently smooth, and thus, we need the stochastic forcing to be sufficiently smooth, too. Throughout this work, we will hence focus on trace-class Wiener processes. In the final Section 2.4, we remark on how to deal with space-time white noise given by the derivative of a cylindrical Wiener process. We expect that our method is still applicable, but the region of validity for the results on stochastic stability might decrease.

2.1 Assumptions on SPDEs and slow manifolds

In the general setting of this chapter, we consider an infinite-dimensional stochastic system in some Hilbert space $\mathcal{H}$ given by the following SPDE

$$du \;=\; \mathcal{A}^\varepsilon(u)\,dt + dW_\varepsilon \;=\; \mathcal{L}^\varepsilon u\,dt + \mathcal{F}^\varepsilon(u)\,dt + dW_\varepsilon. \tag{2.1}$$

Here, $\mathcal{A}^\varepsilon$ denotes a nonlinear differential operator, which might depend on some parameter $\varepsilon > 0$. We split $\mathcal{A}^\varepsilon$ into its linear part $\mathcal{L}^\varepsilon$ and remaining nonlinear terms $\mathcal{F}^\varepsilon$. Moreover, let W_ε be a $\mathcal{Q}_\varepsilon$-Wiener process in the underlying Hilbert space $\mathcal{H}$, where $\mathcal{Q}_\varepsilon$ is a symmetric operator and $(e_k)_{k\in\mathbb{N}}$ forms a complete $\mathcal{H}$-orthonormal basis of eigenfunctions with corresponding eigenvalues α_k^2, i.e., $\mathcal{Q}_\varepsilon e_k = \alpha_k^2 e_k$. It is well known that W_ε is given as the following Fourier series, cf. Da Prato and Zabzcyck [DPZ92b],

$$W_\varepsilon(t) \;=\; \sum_{k\in\mathbb{N}} \alpha_k \beta_k(t) e_k(\cdot)$$

for a family of independent $\mathbb{R}$-valued standard Brownian motions $\{\beta_k(t)\}_{k\in\mathbb{N}}$. For the sake of simplicity, we drop all dependencies on the parameter ε in the remainder (as already carried out with α_k and e_k in the construction of the Wiener process).

We assume that solutions to (2.1) are unique and define a sufficiently smooth stochastic process in $\mathcal{H}$. See, for instance, Appendix C for details on existence, uniqueness, and regularity of solutions to the stochastic Cahn–Hilliard equation. Moreover, we assume that solutions to the stochastic PDE (2.1) can be approximated via some ansatz functions $u^h(t,x) := u(t,x;h)$ for some time-dependent coordinate $h \in \mathbb{R}^N$ indexing the position on the slow manifold. The justification of this approximation can be inferred either from numerical simulations or known properties of the deterministic counterpart of (2.1), e.g., symmetry properties or given shape dynamics, see [CG18]. We collect all ansatz functions u^h in a slow manifold $\mathcal{M}$.

Definition 2.1 (Slow Manifold)**.**
For some set $\mathcal{O} \subset \mathbb{R}^N$ of admissible parameters, we define the *slow manifold*

$$\mathcal{M} := \left\{ u^h \,:\, h \in \mathcal{O} \right\}.$$

We assume that the map $h \mapsto u^h$ defines a C^3-parametrization of $\mathcal{M}$. We denote the j-th partial derivative of u^h with respect to h_j by u^h_j; second and third derivatives, accordingly. Furthermore, we suppose that $\mathcal{M}$ is non-degenerate and defines an N-dimensional manifold.

In order to derive the effective dynamics on the slow manifold, we introduce the concept of Fermi coordinates in a small tubular neighborhood of $\mathcal{M}$. This concept was first used in [CP89, Fun95]. If the solution u is sufficiently close to the slow manifold, we find a unique $\tilde{h} \in \mathcal{O}$ such that $\operatorname{dist}(u, \mathcal{M}) = \inf_{h\in\mathcal{O}} \|u - u^h\| = \|u - u^{\tilde{h}}\|$. Differentiating the map $\Phi(h) = \frac{1}{2}\|u - u^h\|^2$ with respect to all variables h_i, we also see that

$$\langle u - u^{\tilde{h}}, u^{\tilde{h}}_i \rangle = 0 \quad \forall i = 1, \dots, N.$$

This can be interpreted as the vector $v = u - u^{\tilde{h}}$ being orthogonal to the tangent space of $\mathcal{M}$ in $u^{\tilde{h}}$. The Hessian matrix of the map Φ,

$$(H_\Phi(h))_{i,j} = \langle u^h_i, u^h_j \rangle - \langle u - u^h, u^h_{ij} \rangle,$$

is closely related to the first fundamental form P of the manifold $\mathcal{M}$ defined by $P_{ij} = \langle u^h_i, u^h_j \rangle$. Since the vectors u^h_i, $i = 1, \dots, N$, form a basis of the tangent space $\mathcal{T}_{u^h}\mathcal{M}$, we can find for any $w \in \mathcal{T}_{u^h}\mathcal{M}$ a vector $\alpha = (\alpha_1, \dots, \alpha_N)$ such that $w = \sum \alpha_i u^h_i$. Thus we obtain

$$\|w\|^2 = \sum_{i,j=1}^N \alpha_i \alpha_j \langle u^h_i, u^h_j \rangle = \alpha^\intercal P \alpha,$$

and therefore, the fundamental form P is positive definite. If the distance $\|u - u^h\|$ is sufficiently small, we see that the Hessian $H_\Phi(h)$ is positive definite as well, and hence, Φ only obtains minima close to the slow manifold $\mathcal{M}$.

Definition 2.2 (Fermi coordinates)**.**
Let $u(t)$ be the unique solution to (2.1). We define the pair of coordinates $(h(t), v(t)) \in \mathcal{O} \times \mathcal{H}$ such that

$$u(t) = u^{h(t)} + v(t), \quad \langle u^{h(t)}_i, v(t) \rangle = 0 \quad \text{for} \quad i = 1, \dots, N, \tag{2.2}$$

as *Fermi coordinates* or *tubular coordinates* of $u(t)$.

For some fixed point $u(t)$, minimizing the function Φ always leads to a pair $(h(t), v(t))$ satisfying Definition 2.2 unless $u^{h(t)}$ hits the boundary of $\mathcal{M}$. In order to guarantee that the Fermi coordinates $h(t)$ and $v(t)$ depend smoothly on t, we need to assure local uniqueness.

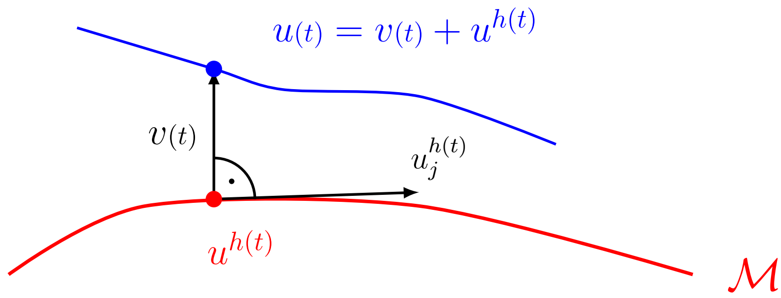

Figure 2.1.: Splitting of the solution u into its tangential and orthogonal part

Therefore, we work in a sufficiently small tubular neighborhood of $\mathcal{M}$ such that the projection (2.2) is well-defined. For local uniqueness of the Fermi coordinates, we refer to Theorem 3.8 of [DH94]. This is a standard result in differential geometry.
Later in Subsection 2.2.2, we present another method to ensure that the Fermi coordinates are locally well-defined. There we will see that—under Lipschitz conditions on the coefficients—local solutions to the stochastic ODE governing the motion of the shape variable h naturally lead to tubular coordinates in a small neighborhood of $\mathcal{M}$.

Remark 2.3 (Approximation of the tangent space)**.**
In some applications, however, it is useful to approximate the tangent space $\mathcal{T}_{u^h}\mathcal{M}$ by the span of some functions E_k^h for $k \in \{1, \dots, N\}$ (see Definition 2.4). Due to slow motion, for instance, the residual $\mathcal{A}(u^h)$ is typically very small, and thus, by differentiating with respect to h_k, we expect that $\mathrm{D}\mathcal{A}(u^h)u_k^h$ is small as well. Therefore, the functions u_k^h can be seen as good approximations of the eigenfunctions corresponding to the small eigenvalues of $\mathrm{D}\mathcal{A}(u^h)$, the linearization of $\mathcal{A}$ at the ansatz function u^h. In some applications, the exact eigenfunctions E_k^h are known, and it is more convenient to work with them. For example, in the stability analysis, we need that eigenvalues corresponding to eigenfunctions orthogonal to the tangent space are negative and uniformly bounded away from zero. See Section 2.3 for more details. For this reason, we define an approximation of the tangent space $\mathcal{T}_{u^h}\mathcal{M}$.

Definition 2.4 (Approximate tangent space)**.**
For $k \in \{1, \dots, N\}$, the functions E_k^h denote *approximations of the tangent vectors* $u_k^h \in \mathcal{T}_{u^h}\mathcal{M}$. We assume that they satisfy the following properties:

i) the map $\mathcal{O} \ni h \mapsto E_k^h \in \mathcal{H}$ is smooth (at least C^2). We denote the partial derivative of E_k^h with respect to h_j by $E_{k,j}^h$, and second derivatives $E_{k,ij}^h$, accordingly.

ii) the linear space spanned by the functions $E_1^h, \dots, E_N^h$ is non-degenerate, that is,
$$\dim \operatorname{span}\left\{E_i^h \,:\, i = 1, \dots, N\right\} = N \quad \forall h \in \mathcal{O}.$$

iii) the function E_k^h serves as a good approximation of the function u_k^h, where u^h is given by Definition 2.1, i.e., the quantity $\|E_k^h - u_k^h\|$ is very small.

In this case, the pair $(h(t), v(t)) \in \mathcal{O} \times \mathcal{H}$ such that
$$u(t) = u^{h(t)} + v(t), \quad \langle E_k^{h(t)}, v(t)\rangle = 0 \quad \text{for} \quad k = 1, \dots, N \tag{2.3}$$
will also be referred to as *Fermi coordinates* of $u(t)$.

In the remainder, we suppose that the shape variable h performs an N-dimensional diffusion process given by

$$dh \;=\; b(h,v)\,dt + \langle \sigma(h,v), dW\rangle, \tag{2.4}$$

for some vector field $b : \mathcal{O}\times\mathcal{H} \to \mathbb{R}^N$ and some diffusion $\sigma : \mathcal{O}\times\mathcal{H} \to \mathcal{H}^N$ (cf. Theorem 2.6 for the exact formulas). This assumption will make the computations in the following section feasible. Later in Lemma 2.8, we justify this ansatz and comment on why it is not restrictive for h being a diffusion process.

2.2 Effective dynamics along the slow manifold

The main aim of this paragraph is to identify the drift term $b : \mathcal{O} \to \mathbb{R}^N$ and the diffusion $\sigma : \mathcal{O} \to \mathcal{H}^N$ of the Itô diffusion (2.4), which might also depend on the normal component v, such that u^h is a good approximation of the solution u, cf. (2.3) and (2.4). Since the following calculation is based on the application of Itô formula, we will assume that the Wiener process is trace-class, that is,

$$\mathrm{trace}_{\mathcal{H}}(\mathcal{Q}) \;=\; \sum_{k\in\mathbb{N}} \alpha_k^2 \;=:\; \eta_0 \;<\; \infty.$$

In the derivation of the effective dynamics, we rely on the approximation of the tangent space given by Definition 2.4 and the general Fermi coordinates (2.3). See also Remark 2.3. The adaption to the originally postulated Fermi coordinates of Definition 2.2 is straightforward. We use the Itô formula to differentiate (2.3) with respect to t and obtain

$$du \;=\; dv + \sum_{j=1}^{N} u_j^h\, dh_j + \frac{1}{2}\sum_{i,j=1}^{N} u_{ij}^h\, dh_i\, dh_j.$$

Taking the inner product of this equation with the functions E_k^h in the Hilbert space $\mathcal{H}$ yields for any $k = 1,\dots,N$

$$\langle E_k^h, du\rangle \;=\; \langle E_k^h, dv\rangle + \sum_{j=1}^{N}\langle E_k^h, u_j^h\rangle\, dh_j + \frac{1}{2}\sum_{i,j=1}^{N}\langle E_k^h, u_{ij}^h\rangle\, dh_i\, dh_j. \tag{2.5}$$

Similarly, by multiplying equation (2.1) with E_k^h, we obtain

$$\langle E_k^h, du\rangle \;=\; \langle E_k^h, \mathcal{A}(u^h+v)\rangle\, dt + \langle E_k^h, dW\rangle. \tag{2.6}$$

In the following lemma, we deal with computing the product $dh_i\, dh_j$ for a diffusion process h satisfying (2.4). The proof is quite standard and, for example, can be found in [ABBK15]. For the sake of completeness, we state the proof here in detail.

Lemma 2.5. *Let h be given by the diffusion process (2.4). Then, for any $i,j \in \{1,\dots,N\}$, it holds true that*

$$dh_i\, dh_j \;=\; \langle \mathcal{Q}\sigma_i(h), \sigma_j(h)\rangle\, dt.$$

Proof. Since $dW\,dt = 0$ and $dt\,dt = 0$, it is sufficient to consider the term $\langle\sigma_i, dW\rangle\langle\sigma_j, dW\rangle$. Using the series expansion of W together with $d\beta_k(t)\,d\beta_l(t) = \delta_{kl}\,dt$, we obtain by Parceval's identity

$$\begin{aligned}\langle\sigma_i, dW\rangle\langle\sigma_j, dW\rangle \;&=\; \sum_{k,l\in\mathbb{N}} \alpha_k\,\alpha_l\,\langle\sigma_i, e_k\rangle\langle\sigma_j, e_l\rangle\, d\beta_k\, d\beta_l \\ &=\; \sum_{k\in\mathbb{N}} \alpha_k^2\langle\sigma_i, e_k\rangle\langle\sigma_j, e_k\rangle\, dt \;=\; \langle\mathcal{Q}\sigma_i, \sigma_j\rangle\, dt.\end{aligned}$$

□

As a next step, we combine equations (2.5) and (2.6) and utilize Lemma 2.5. This yields directly that

$$\begin{aligned}\sum_{j=1}^{N}\langle E_k^h, u_j^h\rangle\, dh_j \;&=\; -\,\langle E_k^h, dv\rangle - \frac{1}{2}\sum_{i,j=1}^{N}\langle E_k^h, u_{ij}^h\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt \\ &\quad + \langle E_k^h, \mathcal{A}(u^h+v)\rangle\, dt + \langle E_k^h, dW\rangle.\end{aligned} \tag{2.7}$$

We need to eliminate the term involving the differential dv. We apply the Itô formula to the orthogonality condition $\langle E_k^h, v\rangle = 0$ and arrive at

$$\begin{aligned}\langle E_k^h, dv\rangle \;&=\; -\,\langle dE_k^h, v\rangle - \langle dE_k^h, dv\rangle \\ &=\; -\sum_{j=1}^{N}\langle E_{k,j}^h, v\rangle\, dh_j - \frac{1}{2}\sum_{i,j=1}^{N}\langle E_{k,ij}^h, v\rangle\, dh_i\, dh_j \\ &\quad\; -\sum_{j=1}^{N}\langle E_{k,j}^h, dv\rangle\, dh_j - \frac{1}{2}\sum_{i,j=1}^{N}\langle E_{k,ij}^h, dv\rangle\, dh_i\, dh_j \\ &=\; -\sum_{j=1}^{N}\langle E_{k,j}^h, v\rangle\, dh_j - \frac{1}{2}\sum_{i,j=1}^{N}\langle E_{k,ij}^h, v\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt - \sum_{j=1}^{N}\langle E_{k,j}^h, dv\rangle\, dh_j.\end{aligned} \tag{2.8}$$

Note that third-order differentials are always zero and were therefore neglected in the previous calculation. We utilize now that $dv = du - du^h$ by definition of the Fermi coordinates (h, v), and the fact that $dW\, dt = 0$ and $dt\, dt = 0$. We derive

$$\begin{aligned}-\sum_{j=1}^{N}\langle E_{k,j}^h, dv\rangle\, dh_j \;&=\; -\sum_{j=1}^{N}\langle E_{k,j}^h, du\rangle\, dh_j + \sum_{j=1}^{N}\langle E_{k,j}^h, du^h\rangle\, dh_j \\ &=\; -\sum_{j=1}^{N}\langle E_{k,j}^h, \mathcal{Q}\sigma_j\rangle\, dt + \sum_{i,j=1}^{N}\langle E_{k,j}^h, u_i^h\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt.\end{aligned} \tag{2.9}$$

By plugging (2.9) into (2.8), we obtain

$$\langle E_k^h, dv\rangle \;=\; -\sum_{j=1}^{N}\langle E_{k,j}^h, v\rangle\, dh_j + \sum_{i,j=1}^{N}\left[\langle E_{k,j}^h, u_i^h\rangle - \frac{1}{2}\langle E_{k,ij}^h, v\rangle\right]\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt - \sum_{j=1}^{N}\langle E_{k,j}^h, \mathcal{Q}\sigma_j\rangle\, dt.$$

This implies together with (2.7)

$$\begin{aligned}\sum_{j=1}^{N}\left[\langle E_k^h, u_j^h\rangle - \langle E_{k,j}^h, v\rangle\right] dh_j = &\sum_{i,j=1}^{N}\left[\frac{1}{2}\langle E_{k,ij}^h, v\rangle - \langle E_{k,j}^h, u_i^h\rangle - \frac{1}{2}\langle E_k^h, u_{ij}^h\rangle\right]\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt \\ &+ \sum_{j=1}^{N}\langle E_{k,j}^h, \mathcal{Q}\sigma_j\rangle\, dt + \langle E_k^h, \mathcal{A}(u^h+v)\rangle\, dt + \langle E_k^h, dW\rangle.\end{aligned} \tag{2.10}$$

With the intention of simplifying the notation for the following computations, we write the left-hand side of (2.10) as $A(h, v)\cdot dh$, where the matrix $A(h, v)$ is given by

$$A_{kj}(h, v) \;:=\; \langle E_k^h, u_j^h\rangle - \langle E_{k,j}^h, v\rangle.$$

Moreover, in order to solve equation (2.10) for dh and thereby obtain the exact formula for the ansatz (2.4), we need to assume that the matrix $A(h, v)$ is invertible, which is an assumption on the parametrization of the slow manifold $\mathcal{M}$, the approximate tangent vectors E_k^h, and the smallness of v in $\mathcal{H}$. For a later analysis of the stochastic ODE governing the motion of h, it is also convenient to have estimates on the inverse matrix. We summarize this in our first metatheorem.

Definition & Metatheorem 1 (The matrix $A(h,v)$ and invertibility)**.**
We define the matrix $A(h,v) \in \mathbb{R}^{N\times N}$ by

$$A_{kj}(h,v) \coloneqq \langle E^h_k, u^h_j\rangle - \langle E^h_{k,j}, v\rangle.$$

As long as the solution u to the stochastic PDE (2.1) lies in a small tubular neighborhood of $\mathcal{M}$, given by $\|v\|$ being sufficiently small and $u^h \in \mathcal{M}$, we assume that the matrix $A(h,v)$ is invertible. Along with the invertibility of $A(h,v)$, we need estimates of the inverse $A^{-1}(h,v)$.

Plugging the diffusion process (2.4) into equation (2.10) yields directly that the diffusion term σ is given by

$$\sum_{j=1}^N A_{kj}(h,v)\,\sigma_j(h) \;=\; E^h_k.$$

For the drift term b we obtain

$$\begin{aligned}\sum_{j=1}^N A_{kj}(h,v) b_j(h) \;=&\; \sum_{i,j=1}^N \left[\frac{1}{2}\langle E^h_{k,ij}, v\rangle - \langle E^h_{k,j}, u^h_i\rangle - \frac{1}{2}\langle E^h_k, u^h_{ij}\rangle\right]\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\\ &+ \sum_{j=1}^N \langle E^h_{k,j}, \mathcal{Q}\sigma_j\rangle + \langle E^h_k, \mathcal{A}(u^h+v)\rangle.\end{aligned}$$

Using the invertibility of $A(h,v)$, we finally obtain expressions for b and σ.

Theorem 2.6 (Effective dynamics on $\mathcal{M}$)**.**
Suppose that the solution u to (2.1) can be decomposed into the Fermi coordinates (2.3) and that h is given by the diffusion process (2.4). Then, under the assumptions of Metatheorem 1, the drift b and diffusion σ are given by

$$\sigma_r(h) \;=\; \sum_{i=1}^N A_{ri}(h,v)^{-1} E^h_i \tag{2.11}$$

and

$$\begin{aligned}b_r(h) \;=&\; \sum_{i=1}^N A_{ri}(h,v)^{-1}\langle E^h_i, \mathcal{A}(u^h+v)\rangle + \sum_{i=1}^N A_{ri}(h,v)^{-1}\sum_j \langle E^h_{i,j}, \mathcal{Q}\sigma_j\rangle\\ &+ \sum_{i,j,k=1}^N A_{ri}(h,v)^{-1}\left[\frac{1}{2}\langle E^h_{i,jk}, v\rangle - \langle E^h_{i,j}, u^h_k\rangle - \frac{1}{2}\langle E^h_i, u^h_{jk}\rangle\right]\langle \mathcal{Q}\sigma_j, \sigma_k\rangle.\end{aligned} \tag{2.12}$$

Remark 2.7. The first summand in (2.12) is the only term that would survive in the deterministic case. All the other terms appear due to stochastic calculus. Therefore, as we will see later, we need estimates of higher order derivatives that, in general, were not considered in the literature treating the deterministic problems. The assumption of h being a diffusion process was advantageous in the previous calculation since we could control the nonlinear terms $dh_i\,dh_j$ appearing due to Itô calculus (cf. Lemma 2.5).

The ansatz (2.4) combined with equations (2.11) and (2.12) gives the exact stochastic equation for the motion along the manifold $\mathcal{M}$. Based on the different kind of models, it would be useful to approximate this equation in terms of the parameter ε. This will be carried out in more detail for the Cahn–Hilliard and Allen–Cahn equation in subsequent chapters. See also Section 2.2.3 for a comparison to the dynamics given by the projection onto the tangent space of the slow manifold $\mathcal{M}$.

To complete the picture of the exact dynamics, we give the stochastic PDE for the „normal“ component v. Note that both b and σ, as well as the matrix A, depend on v. Differentiating $v = u - u^h$ leads with Itô calculus to

$$\begin{aligned} dv \;&=\; du - du^h \;=\; \mathcal{A}(u)\,dt + dW - \sum_{j=1}^{N} u_j^h\,dh_j - \frac{1}{2}\sum_{i,j=1}^{N} u_{ij}^h\,dh_i\,dh_j \\ &=\; \mathcal{A}(u^h+v)\,dt + dW - \sum_{j=1}^{N} u_j^h\,dh_j - \frac{1}{2}\sum_{i,j=1}^{N} u_{ij}^h\,\langle \mathcal{Q}\sigma_i,\sigma_j\rangle\,dt. \end{aligned} \tag{2.13}$$

2.2.1 Justification of diffusion

In the ansatz (2.4), we made the assumption that h is given by an N-dimensional diffusion process

$$dh \;=\; b(h,v)\,dt + \langle \sigma(h,v), dW\rangle.$$

The following lemma shows that this ansatz is indeed justifiable.

Lemma 2.8 (Justification of diffusion)**.**
Consider the pair of functions (h,v) as solutions to the system given by (2.13) and the ansatz (2.4), where b and σ are given by (2.11) and (2.12). Furthermore, assume that Metatheorem 1 holds true, i.e., the matrix $A(h,v)$ is invertible for all times, and that the initial condition $u(0) = u^{h(0)} + v(0)$ satisfies $\langle v(0), E_k^{h(0)}\rangle = 0$ for all $k = 1,\dots,N$.
Then, $u = u^h + v$ solves (2.1) with $\langle E_k^h, v\rangle = 0$ for $k = 1,\dots,N$.

Proof. In order to show that $u = u^h + v$ solves (2.1), one basically reverses the calculation of this section that lead to the definitions (2.11) and (2.12) of the coefficients σ and b. The orthogonality condition follows from $d\langle E_k^h, v\rangle = 0$ for all $k \in \{1,\dots,N\}$ and the assumption that $\langle v(0), E_k^{h(0)}\rangle = 0$. In more detail, we have

$$\begin{aligned} d\langle E_k^h, v\rangle \;&=\; \langle E_k^h, dv\rangle + \langle dE_k^h, v\rangle + \langle dE_k^h, dv\rangle \\ &=\; \langle E_k^h, dv\rangle + \langle dE_k^h, v\rangle + \langle dE_k^h, du\rangle - \langle dE_k^h, du^h\rangle \\ &=\; \langle \mathcal{A}(u), E_k^h\rangle\,dt + \langle E_k^h, dW\rangle + \sum_j \Big[\langle E_{k,j}^h, v\rangle - \langle u_j^h, E_k^h\rangle\Big]\,dh_j \\ &\qquad + \sum_{i,j}\Big[\frac{1}{2}\langle E_{k,ij}^h, v\rangle - \frac{1}{2}\langle u_{ij}^h, E_k^h\rangle - \langle E_{k,j}^h, u_i^h\rangle\Big]\,\langle \mathcal{Q}\sigma_i,\sigma_j\rangle\,dt + \sum_j \langle E_{k,j}^h, \mathcal{Q}\sigma_j\rangle\,dt. \end{aligned}$$

First of all, we extract the dW–terms and see that

$$E_k^h - \sum_j \Big[\langle u_j^h, E_k^h\rangle - \langle E_{k,j}^h, v\rangle\Big]\,\sigma_j \;=\; E_k^h - \sum_j A_{kj}\sigma_j \;\overset{(2.11)}{=}\; 0.$$

Secondly, we consider the drift term:

$$\begin{aligned} &\sum_{i,j}\Big[\frac{1}{2}\langle E_{k,ij}^h, v\rangle - \langle E_{k,j}^h, u_i^h\rangle - \frac{1}{2}\langle u_{ij}^h, E_k^h\rangle\Big]\,\langle \mathcal{Q}\sigma_i,\sigma_j\rangle + \sum_j \langle E_{k,j}^h, \mathcal{Q}\sigma_j\rangle \\ &\qquad + \langle \mathcal{A}(u^h+v), E_k^h\rangle - \sum_j \Big[\langle u_j^h, E_k^h\rangle - \langle E_{k,j}^h, v\rangle\Big]\,b_j \;\overset{(2.12)}{=}\; 0. \end{aligned}$$

This completes the proof that h is indeed a semimartingale. □

2.2.2 Local existence of the Fermi coordinates

We can use Lemma 2.8 to show that the Fermi coordinates from Definition 2.4 are well-defined. For u being the unique solution to (2.1) and $h \in \mathcal{O}$, we define

$$\tilde{b}(h) := b(h, u - u^h) \quad \text{and} \quad \tilde{\sigma}(h) := \sigma(h, u - u^h).$$

As long as $\|u - u^h\|$ is sufficiently small, we typically have that $\tilde{b}$ and $\tilde{\sigma}$ are locally Lipschitz continuous in h. In fact, by the explicit formulas of Theorem 2.6, we see that the functions $\tilde{b}$ and $\tilde{\sigma}$ depend on various derivatives of the approximate tangent vectors E_i^h and the ansatz function u^h, the operator $\mathcal{A}$, and the inverse of the matrix $A(h, u - u^h)$. If these quantities—for $\|u - u^h\|$ sufficiently small—depend smoothly on h, one can easily derive the Lipschitz continuity of $\tilde{b}$ and $\tilde{\sigma}$. For a detailed analysis, we refer to subsequent applications.

Using the Lipschitz continuity of the coefficients, we then find a unique local solution to the SDE (cf. Appendix Theorem B.12)

$$dh = \tilde{b}(h)\,dt + \langle \tilde{\sigma}(h), dW\rangle.$$

By defining $v := u - u^h$, Lemma 2.8 implies that the pair (h, v) indeed defines tubular coordinates in a small neighborhood of $\mathcal{M}$.

Remark 2.9. Note that if the curvature of the manifold $\mathcal{M}$ is large, the orthogonal projection onto $\mathcal{M}$ might result in a different (non-continuous) curve h. With the method presented here, we obtain a unique smooth curve that defines admissible Fermi coordinates but does not necessarily minimize the distance to $\mathcal{M}$.

2.2.3 Projection onto the slow manifold

We can interpret the effective dynamics given by Theorem 2.6 in terms of the projection onto the slow manifold $\mathcal{M}$. We denote the tangent space of $\mathcal{M}$ at $w \in \mathcal{M}$ by $\mathcal{T}_w\mathcal{M}$. Moreover, let $P_w : \mathcal{H} \to \mathcal{T}_w\mathcal{M}$ be the projection of the Hilbert space $\mathcal{H}$ onto $\mathcal{T}_w\mathcal{M}$, which is well-defined as $\mathcal{T}_w\mathcal{M}$ is a finite-dimensional and thus closed subspace of $\mathcal{H}$. For u being a solution to the stochastic PDE

$$du = \mathcal{A}(u)\,dt + dW = \mathcal{L}u\,dt + \mathcal{F}(u)\,dt + dW,$$

we compare the effective equations derived in Theorem 2.6 with the flow on $\mathcal{M}$ generated by the projection onto the slow manifold, that is,

$$dw = P_w\mathcal{A}(w)\,dt + P_w \circ dW. \tag{2.14}$$

To make the following computations easier, we represent the flow on $\mathcal{M}$ as a Stratonovich SDE (cf. Appendix Remark B.11). In Definition 2.1, we assumed that the slow manifold $\mathcal{M}$ can be described by a single smooth chart $\mathcal{O} \ni \bar{h} \mapsto u^{\bar{h}} \in \mathcal{M}$. Note that thereby the tangent space $\mathcal{T}_{u^{\bar{h}}}\mathcal{M}$ at $u^{\bar{h}}$ is given by the span of the functions $u_i^{\bar{h}}$, the partial derivatives of $u^{\bar{h}}$ with respect to $\bar{h}_i$. Thus for $z \in \mathcal{H}$, the projection onto $\mathcal{T}_{u^{\bar{h}}}\mathcal{M}$ is given by $P_{u^{\bar{h}}}z = \sum_{i=1}^N \langle z, u_i^{\bar{h}}\rangle\, u_i^{\bar{h}}$. Writing the solution of (2.14) as $w = u^{\bar{h}(t)}$, we obtain

$$\sum_{i=1}^N u_i^{\bar{h}} \circ d\bar{h}_i = dw = \sum_{i=1}^N \langle \mathcal{A}(u^{\bar{h}}), u_i^{\bar{h}}\rangle\, u_i^{\bar{h}}\,dt + \sum_{i=1}^N \langle u_i^{\bar{h}}, \circ\, dW\rangle\, u_i^{\bar{h}}.$$

We denote by $S(\bar{h})$ the induced metric of $\mathcal{M}$ on $\mathcal{O} \subset \mathbb{R}^N$, i.e., $S(\bar{h}) := (\langle u_i^{\bar{h}}, u_j^{\bar{h}}\rangle)_{N\times N}$. Note that the matrix $S(\bar{h})$ is invertible, since the functions $u_i^{\bar{h}}$ span the tangent space.

Solving for $d\bar{h}$, we find that

$$d\bar{h}_r = \sum_{j=1}^{N} S_{rj}^{-1}(\bar{h})\langle \mathcal{A}(u^{\bar{h}}), u_j^{\bar{h}}\rangle \, dt + \sum_{j=1}^{N} S_{rj}^{-1}(\bar{h})\langle u_j^{\bar{h}}, \circ \, dW\rangle. \tag{2.15}$$

Equation (2.15) gives the exact dynamics of the projection onto the slow manifold $\mathcal{M}$. Redoing the derivation that led to the full effective dynamics given in Theorem 2.6 in the Stratonovich sense, it is easily seen that we obtain a similar version of (2.15), which also depends on the normal component v, namely

$$dh_r = \sum_{j=1}^{N} A_{rj}^{-1}(h,v)\langle \mathcal{A}(u^h + v), u_j^h\rangle \, dt + \sum_{i=1}^{N} A_{rj}^{-1}(h,v)\langle u_j^h, \circ \, dW\rangle. \tag{2.16}$$

Here, the induced metric $S(h)$ is replaced by the matrix $A(h,v) = (\langle u_k^h, u_j^h\rangle - \langle u_{kj}^h, v\rangle)_{N\times N}$. Also, note that most of the terms appearing in the definition (2.12) of the drift term b arise from Itô–Stratonovich corrections. To make the comparison between the full dynamics and the exact projection more vivid, we stated the full dynamics (2.16) without any approximation of the tangent space (cf. Remark 2.3). By setting the normal component $v = 0$ in equation (2.16), we obtain exactly the projection (2.15) onto the slow manifold $\mathcal{M}$. Hence, up to times where the normal component v stays sufficiently small and the Fermi coordinates are uniquely defined, we expect that the effective dynamics of the shape variable h is well approximated by the flow on $\mathcal{M}$ generated by the projection onto the tangent space. In fact, by choosing an even smaller time scale, we observe that (2.15) is dominated by the projection of the Wiener process onto the slow manifold. See Sections 3.2.4 and 5.3 for details regarding the stochastic Cahn–Hilliard and Allen–Cahn equation.

2.3 Stochastic stability

In the preceding section, we have seen that—for defining a coordinate system around $\mathcal{M}$, invertibility of the matrix $A(h,v)$, deriving the exact equation on $\mathcal{M}$, and so on—it is crucial that the residual error $\|v(t)\|$ stays small for very long times. Therefore, we deal with establishing stochastic stability. Since we computed the stochastic equation for the shape variable h in Theorem 2.6, we aim to show that the error of approximation $v = u - u^h$ stays small for long times with high probability. Motivated by the stability analysis of the one-dimensional Cahn–Hilliard and two-dimensional Allen–Cahn equation in [ABBK15, ABK12], we introduce a useful method based on a stochastic differential inequality of the type

$$d\|v\|^2 \le \Big[K_\varepsilon - a_\varepsilon\|v\|^2\Big] \, dt + \langle \mathcal{O}_{\mathcal{H}}(\|v\|), dW\rangle \tag{2.17}$$

for some ε-dependent constants $a_\varepsilon, K_\varepsilon > 0$. Recall that also the Wiener process W depends on the small parameter ε. To be more precise, we will use the notation of (2.17) for the following inequality in integral form:

$$\|v(t)\|^2 + a_\varepsilon \int_0^t \|v(s)\|^2 \, \mathrm{d}s \le \|v(0)\|^2 + K_\varepsilon \, t + \int_0^t \langle \mathcal{O}_{\mathcal{H}}(\|v(s)\|), \mathrm{d}W(s)\rangle.$$

In the remainder of this section, we present a general guideline to establish such an inequality and, once derived, show how to use (2.17) to prove stochastic stability (Theorem 2.14). Recall that by (2.13), the stochastic PDE for the residual $v = u - u^h$ is given by

$$dv = \Big[\mathcal{A}(u^h) + \mathcal{A}^h v + \mathcal{N}^h(v)\Big] \, dt + dW - \sum_{j=1}^{N} u_j^h \, dh_j - \frac{1}{2}\sum_{i,j=1}^{N} u_{ij}^h \langle \mathcal{Q}\sigma_i, \sigma_j\rangle \, dt. \tag{2.18}$$

Here, we expanded $\mathcal{A}(u^h+v)$ via Taylor expansion into a residual term $\mathcal{A}(u^h)$, the linearization $\mathcal{A}^h$ at an ansatz function $u^h \in \mathcal{M}$, and remaining nonlinear terms of higher order $\mathcal{N}^h(v)$, i.e.,

$$\mathcal{A}^h v := \mathcal{L}v + D\mathcal{F}(u^h)v \quad \text{and} \quad \mathcal{N}^h(v) := \mathcal{F}(u^h+v) - \mathcal{F}(u^h) - D\mathcal{F}(u^h)v,$$

where D$\mathcal{F}$ denotes the Fréchet derivative of $\mathcal{F}$.

For the purpose of obtaining the inequality (2.17), let us start by giving a stochastic differential equation for $\|v\|^2 = \|u-u^h\|^2$. With Itô calculus, we observe that

$$d\|v\|^2 = 2\langle v, dv\rangle + \langle dv, dv\rangle.$$

Plugging (2.18) into this relation, we obtain

$$\begin{aligned}\langle dv, dv\rangle &= \Big\langle -\sum_{j=1}^N u_j^h \langle \sigma_j, dW\rangle + dW, -\sum_{j=1}^N u_j^h \langle \sigma_j, dW\rangle + dW\Big\rangle \\ &= \operatorname{trace}(\mathcal{Q})\,dt - 2\sum_{i,j=1}^N \langle u_i^h, \mathcal{Q}\sigma_j\rangle\,dt + \sum_{i,j=1}^N \langle u_i^h, u_j^h\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\,dt,\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\langle v, dv\rangle &= \Big[\langle \mathcal{A}^h v, v\rangle + \langle \mathcal{N}^h(v), v\rangle\Big]\,dt + \langle \mathcal{A}(u^h), v\rangle\,dt \\ &\quad - \sum_{j=1}^N \langle u_j^h, v\rangle\,dh_j - \frac{1}{2}\sum_{i,j=1}^N \langle u_{ij}^h, v\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\,dt + \langle v, dW\rangle.\end{aligned} \tag{2.20}$$

In our first metatheorem towards establishing stability, we deal with the main contribution to the estimate (2.17), namely the linearization $\langle \mathcal{A}^h v, v\rangle$ and the nonlinear part $\langle \mathcal{N}^h(v), v\rangle$. Essential for the following main estimate is a negative upper bound on the quadratic form (Assumption 2.1) and good control of the nonlinearity (Assumption 2.2). Due to slow motion, the residual term $\mathcal{A}(u^h)$ is typically very small (Assumption 2.3).
After that, we will carry out ideas to control the remaining terms of $\langle dv, dv\rangle$ (Metatheorem 3) and $\langle v, dv\rangle$ (Metatheorem 4).

Metatheorem 2 (Main estimate)**.**
As long as $\|v\| < R_\varepsilon$ for some sufficiently small $R_\varepsilon > 0$ and $u^h \in \mathcal{M}$, there exist $a_\varepsilon, C_\varepsilon > 0$ such that

$$\langle \mathcal{A}(u^h+v), v\rangle \le -a_\varepsilon\|v\|^2 + C_\varepsilon. \tag{2.21}$$

Idea of the proof. We give an idea on how to derive this estimate (cf. Assumptions 2.1–2.3). For more details, we refer to its application to our models in later chapters.
Recall that,

$$\mathcal{A}(u^h+v) = \mathcal{A}(u^h) + \mathcal{A}^h v + \mathcal{N}^h(v).$$

In the first step, we need to control the quadratic form $\langle \mathcal{A}^h v, v\rangle$ for v orthogonal to the space spanned by the functions E_i^h for $i \in \{1,\dots,N\}$. Note that span$\{E_i^h\}$ is a good approximation of the tangent space $\mathcal{T}_{u^h}\mathcal{M}$ of $\mathcal{M}$ at u^h, and thus, we need a good negative upper bound to show stability. Let us record this upper bound in the first Assumption 2.1.

Assumption 2.1 (Bound of the quadratic form)**.**
For $v \perp \operatorname{span}\{E_i^h : i = 1,\dots,N\}$, we find $a_\varepsilon > 0$ depending on ε such that

$$\langle \mathcal{A}^h v, v\rangle \le -a_\varepsilon\|v\|^2. \tag{2.22}$$

In upcoming applications, the linearization $\mathcal{A}^h$ will often be a symmetric, selfadjoint operator with compact resolvent. In this case, it is useful to know the eigenvalues of $\mathcal{A}^h v = -\lambda v$ and its corresponding eigenspaces. If for instance, for v orthogonal to $\operatorname{span}\{E_i^h : i = 1, \ldots, N\}$, the eigenvalues are bounded from below by some constant $a_\varepsilon > 0$, the estimate (2.22) would follow automatically.

Many times though, we only know that the space spanned by the functions E_i^h is close to an eigenspace. Essentially, the functions E_i^h are good approximations of u_i^h spanning the tangent space of $\mathcal{M}$ at u^h and typically, due to slow motion, $\mathcal{A}(u^h)$ is exponentially small. Thus, differentiating $\mathcal{A}(u^h)$ with respect to h_i, suggests that the quantity $\mathcal{A}^h u_i^h$ is exponentially small as well. In this sense, we can think of u_i^h being good approximations to the eigenspace corresponding to very small eigenvalues, and hence the functions E_i^h are good approximations of that space as well (cf. Remark 2.3). The following lemma deals with bounding the quadratic form in such scenarios.

Lemma 2.10. *Let $\mathcal{A}^h$ be a symmetric operator and assume that for some normalized $\psi \in \mathcal{H}$ with $\|\mathcal{A}^h\psi\| = \beta$ there exists $\lambda > 0$ such that for all v with $v \perp \psi$*

$$\langle \mathcal{A}^h v, v\rangle \;\le\; -\lambda\|v\|^2.$$

Then, for $w \in \mathcal{H}$ with $|\langle w, \psi\rangle| = \delta\|w\|$ for some $\delta > 0$, it holds true that

$$\langle \mathcal{A}^h w, w\rangle \;\le\; -\lambda\|w\|^2 + \left(\lambda\delta^2 + 2\delta\beta + \delta^2\beta\right)\|w\|^2.$$

Proof. Without loss of generality, let $w \in \mathcal{H}$ be normalized with $\langle w, \psi\rangle = \delta > 0$.
By defining $\tilde{w} := w - \delta\psi$, one readily verifies that $\langle \tilde{w}, \psi\rangle = 0$. By assumption, we thus obtain $\langle \mathcal{A}^h\tilde{w}, \tilde{w}\rangle \le -\lambda\|\tilde{w}\|^2$. Furthermore, we easily compute

$$\|\tilde{w}\|^2 \;=\; \langle w - \delta\psi, w - \delta\psi\rangle \;=\; \|w\|^2 - 2\delta\langle w, \psi\rangle + \delta^2\|\psi\|^2 \;=\; 1 - \delta^2.$$

Using the symmetry of $\mathcal{A}^h$ yields

$$\begin{aligned}\langle \mathcal{A}^h w, w\rangle \;&=\; \langle \mathcal{A}^h\tilde{w}, \tilde{w}\rangle + 2\delta\langle w, \mathcal{A}^h\psi\rangle + \delta^2\langle \mathcal{A}^h\psi, \psi\rangle\\ &\le\; -\lambda\|\tilde{w}\|^2 + 2\delta\beta + \delta^2\beta \;=\; -\lambda + \lambda\delta^2 + 2\delta\beta + \delta^2\beta.\end{aligned}$$

□

After we have dealt with the linearization $\mathcal{A}^h$, we need to control the nonlinearity $\mathcal{N}^h(v)$. By Assumption 2.1, we have a good negative term $-a_\varepsilon\|v\|^2$ from the quadratic form. Depending on the given linear operator $\mathcal{A}^h$, we might as well be able to gain good terms from a brute-force estimate of $\langle \mathcal{A}^h v, v\rangle$.
To make clear what is meant by a brute-force estimate, consider the following example:
In $\mathcal{H} = L^2$, we consider the Laplace operator $\mathcal{A}^h = \Delta$ with Neumann boundary conditions. Integration by parts yields for the quadratic form

$$\langle \Delta v, v\rangle_{\mathcal{H}} \;=\; -\|\nabla v\|_{L^2}^2 \;<\; 0,$$

which is potentially a good negative term for our analysis. In the general case, let us use the notation

$$\langle \mathcal{A}^h v, v\rangle \;\overset{\text{„brute-force“}}{\le}\; -\mathcal{K}_{\mathcal{A}^h}(v) + C_\varepsilon\|v\|^2,$$

where we collect all „good“ estimates in a positive function $\mathcal{K}_{\mathcal{A}^h}$, which depends on the given linear operator $\mathcal{A}^h$. For more clarity, we refer to the stability sections of subsequent chapters. In order to control the nonlinear term, our objective is to absorb as much as possible into the negative terms $-a_\varepsilon\|v\|^2$ and $-\mathcal{K}_{\mathcal{A}^h}(v)$.

Usually, as estimating $\langle \mathcal{N}^h(v), v\rangle$ (Sobolev embedding, Agmon's inequality, interpolation inequalities, and so on) involves products of the good terms and the norm $\|\cdot\|$ itself, we need to assume that $\|v\|$ is sufficiently small. Assumption 2.2 summarizes our strategy.

Assumption 2.2 (Control of the nonlinearity).
We find ε-dependent constants $R_\varepsilon, C_\varepsilon > 0$ and $0 < \lambda < 1$ such that for $\|v\| < R_\varepsilon$ and $u^h \in \mathcal{M}$

$$\langle \mathcal{N}^h(v), v\rangle \;\leq\; \lambda a_\varepsilon \|v\|^2 + \lambda \mathcal{K}_{\mathcal{A}^h}(v) + C_\varepsilon.$$

In order to establish an estimate of the type (2.21), it remains to bound the residual term. As we have mentioned before, due to slow motion and construction of the ansatz functions u^h, the residual error $\mathcal{A}(u^h)$ is typically very small.

Assumption 2.3 (The residual error).
For R_ε and C_ε as in Assumptions 2.1 and 2.2, we have for all $h \in \mathcal{O}$ and $\varepsilon > 0$ sufficiently small

$$\|\mathcal{A}(u^h)\| \;\leq\; \frac{C_\varepsilon}{R_\varepsilon},$$

and thus,

$$|\langle \mathcal{A}(u^h), v\rangle| \;\leq\; \frac{C_\varepsilon}{R_\varepsilon}\|v\| \;\leq\; C_\varepsilon.$$

By combining Assumptions 2.1, 2.2, and 2.3 we obtain the essential estimate from Metatheorem 2, namely

$$\langle \mathcal{A}(u^h + v), v\rangle \;=\; \langle \mathcal{A}(u^h), v\rangle + \langle \mathcal{A}^h v, v\rangle + \langle \mathcal{N}^h(v), v\rangle \;\leq\; -a_\varepsilon\|v\|^2 + C_\varepsilon.$$

This completes the brief guideline on how to establish Metatheorem 2. For more details, we refer to the stability sections of subsequent chapters. □

It remains to bound the remaining terms of the stochastic differential $d\|v\|^2$. We begin with the Itô correction $\langle dv, dv\rangle$, which due to (2.19) is given by

$$\langle dv, dv\rangle \;=\; \mathrm{trace}(\mathcal{Q})\,dt - 2\sum_{i,j=1}^{N} \langle u_i^h, \mathcal{Q}\sigma_j\rangle\,dt + \sum_{i,j=1}^{N} \langle u_i^h, u_j^h\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\,dt.$$

As these terms arise from an Itô correction, the estimate will obviously depend on the given noise strength η_0. Note that for the induced norm of $\mathcal{Q}$ as an operator on $\mathcal{H}$ we always have that (cf. Appendix Definition B.2 and the proof thereafter)

$$\eta_1 \;:=\; \|\mathcal{Q}\|_{L(\mathcal{H})} \;\leq\; \mathrm{trace}(\mathcal{Q}) \;=\; \eta_0.$$

Via the Cauchy–Schwarz inequality, it would be sufficient to have bounds on the gradient $\nabla_h u^h$ with respect to the h-variables and the diffusion σ in order to control $\langle dv, dv\rangle$. Estimating the gradient $\|\nabla_h u^h\|$ in terms of ε does not pose a problem, as we construct u^h explicitly. In (2.11), we showed that σ depends on the normalized functions E_i^h and the inverse of the matrix $A(h, v)$. Therefore, we need to establish bounds on $\|A^{-1}\|$ (Metatheorem 1). The following metatheorem deals with bounding the Itô correction term.

Metatheorem 3 (Control of $\langle dv, dv\rangle$).
For some $A_\varepsilon > 0$, which depends on the norms of $\nabla_h u^h$ and σ, we have

$$|\langle dv, dv\rangle| \;\leq\; A_\varepsilon\,\eta_0.$$

As a next step, we estimate the remaining terms of $\langle v, dv\rangle$, which were not analyzed yet by Metatheorems 2 and 3. These terms are by (2.20) and the ansatz (2.4)

$$\begin{aligned}\langle v, du^h\rangle + \langle v, dW\rangle &= -\sum_{j=1}^{N}\langle u_j^h, v\rangle\, dh_j - \frac{1}{2}\sum_{i,j=1}^{N}\langle u_{ij}^h, v\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt + \langle v, dW\rangle \\ &= \sum_{j=1}^{N}\langle u_j^h, v\rangle\, b_j(h,v)\, dt - \frac{1}{2}\sum_{i,j=1}^{N}\langle u_{ij}^h, v\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt + \Big\langle v - \sum_{j=1}^{N}\langle u_j^h, v\rangle\sigma_j(h,v), dW\Big\rangle.\end{aligned} \tag{2.23}$$

Out of these remaining terms, the second and third summand can be estimated easily, as they only contain derivatives of u^h, the diffusion σ, and the error of approximation v, which is anyways assumed to be small. In order to control the first summand, we need to bound the drift term $b(h,v)$ appearing in the derivation of dh. Recall that by (2.12),

$$\begin{aligned}b_r(h,v) &= \sum_{i=1}^{N} A_{ri}^{-1}\langle E_i^h, \mathcal{A}(u^h+v)\rangle + \sum_{i=1}^{N} A_{ri}^{-1}\sum_{j=1}^{N}\langle E_{i,j}^h, \mathcal{Q}\sigma_j\rangle \\ &\quad + \sum_{i,j,k=1}^{N} A_{ri}^{-1}\left[\frac{1}{2}\langle E_{i,jk}^h, v\rangle - \langle E_{i,j}^h, u_k^h\rangle - \frac{1}{2}\langle E_i^h, u_{jk}^h\rangle\right]\langle \mathcal{Q}\sigma_j, \sigma_k\rangle.\end{aligned}$$

Assumption 2.4 (The drift b)**.**
We find $K_\varepsilon^b > 0$ depending on ε such that for $\|v\| < R_\varepsilon$ and $h \in \mathcal{O}$

$$\|b(h,v)\| \le K_\varepsilon^b.$$

Remark 2.11. Depending on the given operator $\mathcal{A}$, smallness of v in $\mathcal{H}$ might not be sufficient to control $\langle E_i^h, \mathcal{A}(u^h+v)\rangle$ appearing in the drift term b. To handle this term and hence have the SDE (2.4) well-defined, we often need that v is additionally bounded in a suitably chosen normed space. Thereby, we also have to show stochastic stability in that space. We give an idea on how to deal with this at the end of this section. Moreover, if we choose our coordinate system in such a way that $E_i^h = u_i^h$, i.e., we do not rely on approximations of the tangent space, then we are not concerned in bounding the drift b, since the prefactor $\langle u_i^h, v\rangle$ in (2.23) vanishes.

With a bound of the drift term b at hand, one can easily estimate the remainder of $\langle v, dv\rangle$. This leads to the last metatheorem we have to formulate.

Metatheorem 4 (Remainder of $\langle v, dv\rangle$)**.**
We find $B_\varepsilon, c_\varepsilon > 0$ such that for $\|v\| < R_\varepsilon$ and $u^h \in \mathcal{M}$

$$\langle v, du^h\rangle + \langle v, dW\rangle \le B_\varepsilon \eta_0 + \langle \mathcal{O}(c_\varepsilon\|v\|), dW\rangle.$$

For the formulation of Metatheorem 4, we used the standard $\mathcal{O}$-notation. As we frequently rely on this notation throughout this work, we give the definition below.

Definition 2.12 ($\mathcal{O}$-notation)**.**

(i) We say that a scalar term $f(\varepsilon)$ is $\mathcal{O}(g(\varepsilon))$ if there exists a constant $C > 0$ and $\varepsilon_0 > 0$ such that $f(\varepsilon) \le Cg(\varepsilon)$ for $0 < \varepsilon < \varepsilon_0$.

(ii) For a function $f(\varepsilon)$ taking its values in a normed space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$, we write $f(\varepsilon) = \mathcal{O}_{\mathcal{H}}(g(\varepsilon))$ if $\|f(\varepsilon)\|_{\mathcal{H}} = \mathcal{O}(g(\varepsilon))$.

(iii) Exponentially small terms will be denoted by $\mathcal{O}(\exp)$ or $\mathcal{O}_{\mathcal{H}}(\exp)$, respectively.

Finally, we combine the estimates of Metatheorems 2–4 to obtain a stochastic differential inequality of the type (2.17).

Theorem 2.13 (Stochastic differential inequality).
There exist ε–dependent constants $a_\varepsilon, c_\varepsilon, R_\varepsilon > 0$ and $K_\varepsilon(\eta_0)$ depending on η_0 such that for $\|v\| < R_\varepsilon$ and $h \in \mathcal{O}$,

$$d\|v\|^2 \le \Big[K_\varepsilon(\eta_0) - a_\varepsilon \|v\|^2\Big]\, dt + \langle \mathcal{O}(c_\varepsilon \|v\|), dW\rangle.$$

Based on this estimate, we formulate the main theorem on stochastic stability. Here we follow Section 3.2 of [ABBK15].

Theorem 2.14 (Stochastic Stability).
Define the stopping time

$$\tau^* := \inf\Big\{t \in [0, T_\varepsilon \wedge \tau_0] \,:\, \|v(t)\| > R_\varepsilon\Big\},$$

where the deterministic cut-off T_ε satisfies $T_\varepsilon = \varepsilon^{-M}$ for any fixed large $M > 0$ and τ_0 denotes the exit time from $\mathcal{O}$, the set of admissible parameters. Assume that for $t \le \tau^*$

$$d\|v(t)\|^2 \le \Big[K_\varepsilon(\eta_0) - a_\varepsilon \|v(t)\|^2\Big]\, dt + \langle \mathcal{O}(c_\varepsilon \|v(t)\|), dW\rangle, \tag{2.24}$$

where the constants are given in Theorem 2.13. Furthermore, assume that for some $\kappa > 0$

$$\frac{K_\varepsilon(\eta_0) + c_\varepsilon^2 \eta_1}{a_\varepsilon R_\varepsilon^2} = \mathcal{O}(\varepsilon^\kappa) \quad \text{and} \quad \|v(0)\|^2 < \frac{K_\varepsilon(\eta_0) + c_\varepsilon^2 \eta_1}{a_\varepsilon}.$$

Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as ε tends to 0.

To demonstrate the interplay between all the constants appearing in Theorem 2.14, we state the proof from [ABBK15] in full detail.

Proof. Integrating (2.24) yields for all $t \le \tau^*$

$$\|v(t)\|^2 + a_\varepsilon \int_0^t \|v(s)\|^2 \,\mathrm{d}s \le \|v(0)\|^2 + K_\varepsilon(\eta_0) t + \int_0^t \langle \mathcal{O}(c_\varepsilon \|v\|), \mathrm{d}W\rangle.$$

Since the stopping time τ^* is deterministically bounded, we use that by the optional stopping theorem for martingales stopped stochastic integrals still have mean zero (cf. Appendix Theorem B.16). So we obtain

$$\mathbb{E}\|v(\tau^*)\|^2 + a_\varepsilon \,\mathbb{E} \int_0^{\tau^*} \|v(s)\|^2 \,\mathrm{d}s \le \|v(0)\|^2 + K_\varepsilon(\eta_0)\, T_\varepsilon, \tag{2.25}$$

where we utilized that $\tau^* \le T_\varepsilon$ by definition. We extend this to higher powers using Itô calculus. We will denote all constants depending explicitly on p only by C. For $p > 2$ we derive

$$\begin{aligned}
d\|v\|^{2p} &= p\|v\|^{2p-2}\, d\|v\|^2 + \frac{1}{2} p(p-1)\|v\|^{2p-4} d\|v\|^2 d\|v\|^2 \\
&\le p\|v\|^{2p-2}\, d\|v\|^2 + C c_\varepsilon^2 \eta_1 \|v\|^{2p-2}\, dt \\
&\le p\|v\|^{2p-2} \Big[K_\varepsilon(\eta_0) - a_\varepsilon \|v\|^2\Big]\, dt + C c_\varepsilon^2 \eta_1 \|v\|^{2p-2}\, dt + p\|v\|^{2p-2} \langle \mathcal{O}(c_\varepsilon \|v\|), dW\rangle \\
&= -p a_\varepsilon \|v\|^{2p}\, dt + C\Big[K_\varepsilon(\eta_0) + c_\varepsilon^2 \eta_1\Big] \|v\|^{2p-2}\, dt + p\|v\|^{2p-2} \langle \mathcal{O}(c_\varepsilon \|v\|), dW\rangle,
\end{aligned}$$

where we used that $d\|v\|^2 d\|v\|^2 \le C c_\varepsilon^2 \eta_1 \|v\|^2$ by (2.24) and Lemma 2.5.

Hence, for all integers $p > 2$ provided $t \le \tau^*$, we derive by integrating

$$\begin{aligned}&\|v(t)\|^{2p} + pa_\varepsilon \int_0^t \|v(s)\|^{2p}\,\mathrm{d}s \\ &\le \|v(0)\|^{2p} + C\left[K_\varepsilon(\eta_0) + c_\varepsilon^2\eta_1\right]\int_0^t \|v(s)\|^{2p-2}\,\mathrm{d}s + p\int_0^t \langle \mathcal{O}(c_\varepsilon\|v\|^{2p-1}), \mathrm{d}W\rangle.\end{aligned}$$

Thus, by applying the optional stopping theorem to stochastic integrals, we obtain

$$\mathbb{E}\|v(\tau^*)\|^{2p} \le \|v(0)\|^{2p} + C\left[K_\varepsilon(\eta_0) + c_\varepsilon^2\eta_1\right]\mathbb{E}\int_0^{\tau^*} \|v(s)\|^{2p-2}\,\mathrm{d}s \tag{2.26}$$

and

$$a_\varepsilon\int_0^{\tau^*} \|v(s)\|^{2p}\,\mathrm{d}s \le \frac{1}{p}\|v(0)\|^{2p} + C\left[K_\varepsilon(\eta_0) + c_\varepsilon^2\eta_1\right]\mathbb{E}\int_0^{\tau^*} \|v(s)\|^{2p-2}\,\mathrm{d}s. \tag{2.27}$$

For a simpler notation, let us define

$$q = q(\varepsilon, \eta_0, \eta_1) := \frac{K_\varepsilon(\eta_0) + c_\varepsilon^2\eta_1}{a_\varepsilon}.$$

Using that $K_\varepsilon(\eta_0) \le a_\varepsilon q$, we obtain inductively

$$\begin{aligned}\frac{1}{p}\mathbb{E}\|v(\tau^*)\|^{2p} &\overset{(2.26)}{\le} \frac{1}{p}\|v(0)\|^{2p} + C\left[K_\varepsilon(\eta_0) + c_\varepsilon^2\eta_1\right]\mathbb{E}\int_0^{\tau^*} \|v(s)\|^{2p-2}\,\mathrm{d}s \\ &= \frac{1}{p}\|v(0)\|^{2p} + Cqa_\varepsilon\mathbb{E}\int_0^{\tau^*} \|v(s)\|^{2p-2}\,\mathrm{d}s \\ &\overset{(2.27)}{\le} \frac{1}{p}\|v(0)\|^{2p} + Cq\frac{1}{p}\|v(0)\|^{2p-2} + Cq^2a_\varepsilon\mathbb{E}\int_0^{\tau^*} \|v(s)\|^{2p-4}\,\mathrm{d}s \\ &\le \dots \\ &\le Cq^{p-2}\|v(0)\|^4 + Cq^{p-1}a_\varepsilon\mathbb{E}\int_0^{\tau^*} \|v(s)\|^2\,\mathrm{d}s \\ &\overset{(2.25)}{\le} Cq^{p-2}\|v(0)\|^4 + Cq^{p-1}\left[\|v(0)\|^2 + K_\varepsilon T_\varepsilon\right] \le Cq^p + Ca_\varepsilon q^p T_\varepsilon.\end{aligned}$$

Chebychev's inequality finally yields

$$\begin{aligned}\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0) &= \mathbb{P}\left(\|v(\tau^*)\| \ge R_\varepsilon\right) \le R_\varepsilon^{-2p}\,\mathbb{E}\|v(\tau^*)\|^{2p} \\ &\le CR_\varepsilon^{-2p}\left[q^p + aq^pT_\varepsilon\right] = C\left(\frac{q}{R_\varepsilon^2}\right)^p + Ca_\varepsilon\left(\frac{q}{R_\varepsilon^2}\right)^p T_\varepsilon.\end{aligned}$$

Since by assumption $q/R_\varepsilon^2 = \mathcal{O}(\varepsilon^\kappa)$, the statement is proved by choosing p sufficiently large. □

In upcoming applications, it is necessary to extend the stability result to other spaces in order to show that the stochastic ODE governing the motion of the shape variable h is well-defined (cf. Remark 2.11). Let us take a normed space $(\mathcal{K}, \|\cdot\|_\mathcal{K})$ and assume that $\mathcal{H}$ is not continuously embedded into $\mathcal{K}$, .i.e., smallness in $\mathcal{H}$ does not imply smallness in $\mathcal{K}$. In this scenario, we can try to use that we already proved stochastic stability in $\mathcal{H}$ in Theorem 2.14, i.e., $\|v\|_\mathcal{H}$ stays small for polynomial times in ε^{-1} with very high probability. Hence, if we apply the previous method to estimate the stochastic differential $d\|v\|_\mathcal{K}^2$, our estimates can depend on the $\mathcal{H}$-norm of v, which then can be bounded in terms of the radius R_ε given by Theorem 2.14. To be more precise, define for some $r_\varepsilon > 0$ the stopping time

$$\tau_\mathcal{K}(r_\varepsilon) := \inf\Big\{t \in [0, T_\varepsilon \wedge \tau_0] : \|v(t)\|_\mathcal{H} > R_\varepsilon \quad \text{or} \quad \|v(t)\|_\mathcal{K} > r_\varepsilon\Big\}.$$

Up to times $t \le \tau_{\mathcal{K}}(r_\varepsilon)$, our objective is to establish a stochastic differential inequality of the type

$$d\|v\|_{\mathcal{K}}^2 \;\le\; \Big[\tilde{K}_\varepsilon(\eta_0,\eta_{\mathcal{K}}) - \tilde{a}_\varepsilon\,\|v\|_{\mathcal{K}}^2\Big]\,dt + \langle \mathcal{O}(\tilde{c}_\varepsilon\,\|v\|_{\mathcal{K}}), dW\rangle.$$

Then, we can essentially proceed in the same way as in Theorem 2.14 to show stochastic stability in $\mathcal{K}$. Note that the constants $\tilde{K}_\varepsilon$, $\tilde{a}_\varepsilon$, and $\tilde{c}_\varepsilon$ may now depend on the $\mathcal{H}$-radius R_ε. Moreover, since elements of $\mathcal{K}$ are typically more regular than those in $\mathcal{H}$, we need to assume additional regularity of solutions to the SPDE (2.1), and therefore, we also need to have a higher spatial regularity of the Wiener process W. That is why the constant $\tilde{K}_\varepsilon$ depends not only on the noise strength η_0, but also on $\eta_{\mathcal{K}}$ taking care of the additional regularity. For instance, if the norm in $\mathcal{K}$ is given by $\|\cdot\|_{\mathcal{K}} = \|\mathcal{S}^{1/2}\cdot\|_{\mathcal{H}}$ for some selfadjoint operator $\mathcal{S}$, we naturally have to assume that

$$\eta_{\mathcal{K}} \;\coloneqq\; \mathrm{trace}(\mathcal{Q}^{1/2}\mathcal{S}\mathcal{Q}^{1/2}) \;=\; \sum_{k\in\mathbb{N}} \alpha_k^2 \|\mathcal{S}^{1/2}e_k\|_{\mathcal{H}}^2 \;<\; \infty.$$

For more details, we refer to the application to the stochastic Cahn–Hilliard and Allen–Cahn equation in subsequent chapters.

2.4 Singular noise

The method introduced in Section 2.3 relied heavily on the Itô formula, and hence, we needed to assume sufficient smoothness of the stochastic forcing. Although not part of the upcoming applications to the stochastic Cahn–Hilliard and Allen–Cahn equation, we present an approach towards treating rougher noise. For instance, we could drop the assumption that the covariance operator $\mathcal{Q}$ is trace-class and consider a cylindrical Wiener process W, that is, we allow for $Q \equiv I$ and consider space-time white noise. First, we define the stochastic convolution $W_{\mathcal{L}}$ by

$$W_{\mathcal{L}}(t) \;\coloneqq\; \int_0^t e^{-(t-s)\mathcal{L}}\,\mathrm{d}W_s.$$

In this definition, the family $\{e^{t\mathcal{L}}\}_{t\ge 0}$ denotes a C^0-semigroup generated by the linear operator $\mathcal{L}$ (cf. Appendix B.2). Note that $W_{\mathcal{L}}$ is the unique mild solution to the linear equation $du = \mathcal{L}u + dW$. In addition, under suitable assumptions on the semigroup, the stochastic convolution enjoys good regularity properties (L^p-regularity, Hölder continuity, and so on). Moreover, since W is sufficiently small in ε and $\mathcal{L}$ a stable operator, we expect that the stochastic convolution $W_{\mathcal{L}}$ remains small for large time scales. For more details on the topic of regularity, we refer to [DPZ92a, DPL98].

In order to prove stochastic stability, we have to control the residual error $v = u - u^h$, where $u^h \in \mathcal{M}$ denote the ansatz functions in the slow manifold $\mathcal{M}$. Due to the lack of regularity, we cannot apply the Itô formula directly to v. Motivated by [DPD96, BYZ19], we consider instead the difference $\mathcal{Z} \coloneqq u - u^h - W_{\mathcal{L}}$, which has better regularity properties and serves as good approximation of v as $W_{\mathcal{L}}$ is expected to be small. Combining the estimates of $\mathcal{Z}$ and the stochastic convolution $W_{\mathcal{L}}$, one then obtains control of the residual term $u - u^h$. One easily computes that the process $\mathcal{Z}$ satisfies

$$\begin{aligned}\partial_t\mathcal{Z} \;&=\; \mathcal{L}\mathcal{Z} + \mathcal{A}(u^h) + \mathrm{D}\mathcal{F}(u^h)\,[\mathcal{Z} + W_{\mathcal{L}}] + \mathcal{N}(u^h, \mathcal{Z} + W_{\mathcal{L}}) - \partial_t u^h\\ &\approx\; \mathcal{L}^h\mathcal{Z} + \mathrm{D}\mathcal{F}(u^h)W_{\mathcal{L}} + \mathcal{N}(u^h, \mathcal{Z} + W_{\mathcal{L}}) - \partial_t u^h.\end{aligned}$$

For simplicity of presentation, we dropped the term $\mathcal{A}(u^h)$ as it typically is very small due to slow motion.

Applying Itô's formula yields

$$\frac{1}{2}d\|\mathcal{Z}\|^2 = \langle \mathcal{L}^h \mathcal{Z}, \mathcal{Z}\rangle + \langle \mathrm{D}\mathcal{F}(u^h)\mathcal{W}_{\mathcal{L}}, \mathcal{Z}\rangle + \langle \mathcal{N}(u^h, \mathcal{Z} + W_{\mathcal{L}}), \mathcal{Z}\rangle - \langle du^h, \mathcal{Z}\rangle + \frac{1}{2}\langle d\mathcal{Z}, d\mathcal{Z}\rangle.$$

Note that we need the Itô corrections due to u^h. The task at hand is to achieve a similar stochastic differential inequality to the one of Theorem 2.13. For the first term involving the linearized operator $\mathcal{L}^h$ we can rely on the spectral analysis (cf. Assumption 2.1). Note that we do not have $\mathcal{Z} \perp \mathcal{T}_h\mathcal{M}$, but since the stochastic convolution is typically small, the quadratic form is manageable. For example, one could use Lemma 2.10 to establish the spectral estimate plus some small error depending on $W_{\mathcal{L}}$. Similarly, using the smallness of the stochastic convolution and a uniform bound on $\mathrm{D}\mathcal{F}(u^h)$, the second summand is feasible. Opposed to the previous setting in Section 2.3, the nonlinearity $\mathcal{N}(u^h, \mathcal{Z} + W_{\mathcal{L}})$ does not only depend on the new variable $\mathcal{Z}$, but also on the stochastic convolution $W_{\mathcal{L}}$. Hence, we obtain additional terms that need a careful analysis. Naturally, this will influence the radius for which we can prove stochastic stability. Finally, one has to handle the term $\langle du^h, \mathcal{Z}\rangle$ and the Itô correction $\langle d\mathcal{Z}, d\mathcal{Z}\rangle$. Typically, these do not pose a problem. For example, with $\mathcal{Q} \equiv I$ we see that

$$|\langle d\mathcal{Z}, d\mathcal{Z}\rangle| = |\langle du^h, du^h\rangle| \sim \|\sigma(h)\|^2\, dt,$$

where $\sigma = A^{-1}(h)\nabla u^h$ was defined in Theorem 2.6 Equation (2.11). Because of the special structure of the covariance operator, some of the inner products can therefore be computed explicitly.

Following the aforementioned steps, one could treat a more singular noise. We expect that—compared to the smoother cases we study in subsequent chapters—a rougher noise will decrease the region of validity for the results on stochastic stability.

CHAPTER 3

Motion of a single bubble for the stochastic Cahn–Hilliard equation

In this chapter, we apply the methods from the preceding general framework to the stochastic Cahn–Hilliard equation (also known as Cahn–Hilliard–Cook equation) posed on a bounded smooth domain $\Omega \subset \mathbb{R}^d$, where we allow for the space dimensions $d = 2$ and $d = 3$:

$$\begin{cases} \partial_t u \;=\; -\Delta(\varepsilon^2 \Delta u - F'(u)) + \dot{W}(x,t), & x \in \Omega, \\ \partial_\eta u \;=\; \partial_\eta \, \Delta u \;=\; 0, & x \in \partial\Omega. \end{cases} \tag{CH}$$

The Cahn–Hilliard equation serves as phenomenological model for the phase separation and subsequent coarsening of binary alloys. Proposed by John W. Cahn and John E. Hilliard in [CH58, Cah59] at a fixed temperature, it was extended by H. Cook [Coo70] in order to incorporate thermal fluctuations in the form of an additive noise. Here, $\varepsilon > 0$ is a small positive parameter measuring the relative importance of surface energy compared to the bulk free energy, and ∂_η denotes the exterior normal derivative to the boundary $\partial\Omega$. The potential F is assumed smooth with two equal nondegenerate minima at $u = \pm 1$. A typical example is $F(u) = \frac{1}{4}(u^2 - 1)^2$. For simplicity, we focus on this example in many results. Section 3.3.1 is devoted to a general class of potentials with at most polynomial growth at infinity.

The stochastic forcing is given by an additive white in time noise $\partial_t W$. As our methods rely on Itô's formula, we assume that the Wiener process is sufficiently smooth in space, and moreover, sufficiently small in ε, so that it does not destroy the typical patterns in the solutions. The existence and uniqueness of solutions is well-studied (see for example [DPD96, CW01] and Appendix C) and we always assume that, for a given initial condition, we have a unique solution. In addition, as we assume the noise to be smooth in space, the solution is regular in space, too.

A key property of the deterministic Cahn–Hilliard equation is that it forms a gradient flow in the H^{-1}-topology with respect to the Ginzburg–Landau–Wilson energy functional

$$J_\varepsilon(u) \;=\; \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla v|^2 + F(u) \right) \mathrm{d}x. \tag{3.1}$$

For that reason, one can expect that, for $\varepsilon \ll 1$, solutions to (CH) stay mostly near $u = -1$ and $u = +1$, the stable minima of $F(u)$. Therefore, the typical initial condition evolves into a layered function in space. Because of this, as soon as this initial stage is completed, we can think of Ω being split into subdomains on which $u_\varepsilon(\cdot, t)$ takes approximately the constant values -1 and 1, with boundaries ε-localized about an interface $\Gamma_\varepsilon(t)$. The interface is expected to move according to a Hele–Shaw or Mullins–Sekerka problem, where circular shaped droplets are stable stationary solutions of the dynamics.

In our results, we focus on the almost final stage, where the interface is already a single spherical bubble or droplet inside the domain, and thus, the only possible dynamics is given by the translation of the droplet, at least as long as the droplet stays away from the boundary.

An earlier version on results of this chapter was published in [BS20].

Assumptions on spaces and noise

We fix the underlying space $H^{-1}(\Omega)$ with scalar product $\langle \cdot,\cdot\rangle$ and norm $\|\cdot\|$. The standard scalar product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$ or $\langle \cdot,\cdot\rangle_{L^2}$. Moreover, we use $\|\cdot\|_\infty$ for the supremum norm in C^0 or L^∞. As the Cahn–Hilliard equation preserves the total mass, we also consider the subspace $H_0^{-1}(\Omega)$ of the Sobolev space $H^{-1}(\Omega)$ with zero average. Recall that the inner product in $H_0^{-1}(\Omega)$ is given by

$$\langle \psi,\varphi\rangle_{H^{-1}} = \left\langle (-\Delta)^{-1/2}\psi, (-\Delta)^{-1/2}\varphi \right\rangle_{L^2},$$

where $-\Delta$ is the self-adjoint positive operator defined on $L_0^2(\Omega) = \{\varphi \in L^2(\Omega) : \int_\Omega \varphi \,\mathrm{d}x = 0\}$ by the negative Laplacian with Neumann boundary conditions. The stochastic forcing is given by an additive white in time noise $\partial_t W$, where W denotes a $\mathcal{Q}$-Wiener process.

Definition 3.1 (The Wiener process)**.**
Let W be a $\mathcal{Q}$-Wiener process in the underlying Hilbert space $H^{-1}(\Omega)$, $\mathcal{Q}$ a symmetric operator and $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis with corresponding eigenvalues α_k^2 such that

$$\mathcal{Q}e_k = \alpha_k^2 e_k \qquad \text{and} \qquad W(t) = \sum_{k\in\mathbb{N}} \alpha_k \beta_k(t) e_k,$$

for a sequence of independent real-valued standard Brownian motions $\{\beta_k(t)\}_{k\in\mathbb{N}}$, cf. Da Prato and Zabzcyck [DPZ92b].

Although we dropped the index ε in the definition of the Wiener process W for the sake of simplicity, we assume that W, and thus the covariance operator $\mathcal{Q}$, depends on ε. As our analysis relies heavily on the application of Itô's formula, we have to assume that the Wiener process W is sufficiently regular in space. Moreover, we need to guarantee mass conservation of solutions to (CH). In the remainder of this chapter, we rely on the following regularity assumptions of the Wiener process W.

Assumption 3.2 (Regularity of the Wiener process)**.**
We assume that the process W takes its values in H_0^{-1}, that is, it satisfies

$$\int_\Omega W(t,x)\,\mathrm{d}x = 0 \quad \textit{for all} \quad t \geq 0.$$

Furthermore, we suppose that for some constant $c > 0$

$$1)\ \ \|\mathcal{Q}\|_{L(H^{-1})} < c\delta_\varepsilon^2 \qquad \textit{and} \qquad 2)\ \ \sum_{k\in\mathbb{N}} \alpha_k^2 \|e_k\|_{H_\varepsilon^1}^2 < c\delta_\varepsilon^2,$$

where we need the technical condition $\delta_\varepsilon < \varepsilon^{7/2+3d/2}$. Here, $\|\cdot\|_{H_\varepsilon^1}$ is defined as

$$\|e\|_{H_\varepsilon^1} := \left(\int_\Omega \varepsilon^2 |\nabla e(x)|^2 + e(x)^2 \,\mathrm{d}x \right)^{1/2}. \tag{3.2}$$

Note that condition 2) also implies that

$$\mathrm{trace}_{H^{-1}}(\mathcal{Q}) = \sum_{k\in\mathbb{N}} \alpha_k^2 < c\delta_\varepsilon^2.$$

The first assumption on the norm of $\mathcal{Q}$ as an operator in H^{-1} implies that the strength of the noise is bounded by $\mathcal{O}(\delta_\varepsilon)$, while the second one assures additional spatial regularity of the noise. Also, note that the weighted norm $\|\cdot\|_{H_\varepsilon^1}$ is equivalent to the standard H^1-norm with ε-dependent constants.

Later in Section 3.3.1, we extend our analysis to a more general class of nonlinearities. In the three-dimensional case, we need to assume even more regularity of the Wiener process, namely,

$$3)\quad \sum_{k\in\mathbb{N}} \alpha_k^2 \|e_k\|_\eta^2 < c\rho_\varepsilon^2 < \infty,$$

where we define for some $\eta > 0$

$$\|e\|_\eta := \left(\varepsilon^\eta \|\Delta e\|_{L^2}^2 + \|e\|_{H_0^{-1}}^2\right)^{1/2}.$$

The exact sizes of ρ_ε *and* η *will be fixed in Section 3.3.1.*

Remark 3.3. We observe that W is a $\mathcal{Q}$-Wiener process in L^2 if, and only if, W is a $(-\Delta)^{-1/2}\mathcal{Q}(-\Delta)^{-1/2}$-Wiener process in H^{-1}. Since the eigenvalues of $(-\Delta)^{-1}$ behave asymptotically like $k^{-2/d}$, it is easy to check that the condition of $\mathcal{Q}$ being trace-class in H^{-1} includes space-time white noise in spatial dimension $d = 1$, but in our higher dimensional cases space-time white noise is exactly the borderline regularity that we cannot treat.

3.1 The slow manifold

In this section, we collect some important results from the study of the deterministic Cahn–Hilliard equation in higher space dimensions by N. Alikakos and G. Fusco [AF98], which we need throughout this chapter. In our analysis, we rely on the same deterministic slow manifold $\mathcal{M}^\rho$ consisting of droplets with fixed radius $\rho > 0$. The droplet state is given by almost stationary solutions to (CH) (Proposition 3.4). Since the constructed bubble fails the equation or the boundary conditions by an exponentially small term, we have to take care of this deficiency with an exponentially small correction (Theorem 3.7). The slow manifold is then given in Definition 3.8. Afterwards in Section 3.1.2, we discuss the spectrum of the linearized Cahn–Hilliard and Allen–Cahn operator, which is crucial for the stability analysis.

3.1.1 Construction of the droplet state

Supported by the works of Stoth [Sto96] and Alikakos, Bates & Chen [ABC94] on the deterministic problem, the front $\Gamma_\varepsilon(t)$ separating the pure phases moves in the sharp interface limit $\varepsilon \to 0$ according to the geometric evolution law

$$v = b\left[\frac{d\mu}{d\eta}\right]_{\Gamma(t)}, \tag{MS}$$

where

$$\begin{aligned} \Delta\mu &= 0, && x \in \Omega \setminus \Gamma(t),\\ \partial_\eta \mu &= 0, && x \in \partial\Omega,\\ \mu &= \varepsilon a K, && x \in \Gamma(t). \end{aligned}$$

The problem (MS) is referred to as Mullins–Sekerka problem. Here, a and b are constants, K denotes the mean curvature of $\Gamma(t)$ at x, $[\frac{d\mu}{d\eta}]$ is the jump of the normal derivative $\frac{d\mu}{d\eta}$ across $\Gamma(t)$, and v is the normal component of the velocity of $\Gamma(t)$. It is easy to check that a sphere, or more generally a surface consisting of a finite number of non-overlapping spheres contained in Ω, is an equilibrium to the Mullins–Sekerka problem. This suggests that it is fruitful to search for bounded radial stationary solutions to the Cahn–Hilliard equation on the whole space $\mathbb{R}^d$, i.e.,

$$-\Delta\left[\varepsilon^2\Delta u - F'(u)\right] = 0, \quad x \in \mathbb{R}^d.$$

Due to Liouville's theorem a function $u \in C^2(\mathbb{R}^d)$ is such a solution if, and only if, it is radial and satisfies for some constant $\sigma \in \mathbb{R}$

$$\varepsilon^2 \Delta u - F'(u) = \sigma, \quad x \in \mathbb{R}^d.$$

The following proposition concerns the existence of such radial solutions to the rescaled problem $\Delta u - F'(u) = \sigma$. For a detailed proof, we refer to [AF98], Proposition 2.1.

Proposition 3.4 (The droplet, [AF98], Proposition 2.1)**.**
There exist a number $\bar{\rho} > 0$ and smooth functions $\sigma : (\bar{\rho}, \infty) \to \mathbb{R}$, $U^* : [0, \infty) \times (\bar{\rho}, \infty) \to \mathbb{R}$, such that $\sigma(\rho)$ and $u(x, \rho) = U^*(|x|, \rho)$ satisfy for each $\rho \in (\bar{\rho}, \infty)$ the equation

$$\Delta u(x, \rho) - F'(u(x, \rho)) = \sigma(\rho), \quad x \in \mathbb{R}^d. \tag{3.3}$$

Moreover, $U^*(r, \rho)$ is increasing in r and

i) $\sigma(\rho) = C\rho^{-1} + \mathcal{O}(\rho^{-2})$,

ii) $U^*(\rho, \rho) = \mathcal{O}(\rho^{-1})$,

iii) $1 + U^*(0, \rho) = \mathcal{O}(\rho^{-1})$,

iv) $\lim_{r\to\infty} U^*(r, \rho) = \alpha(\rho)$, *where $\alpha(\rho)$ denotes the root near 1 of the equation $F'(u) + \sigma(\rho) = 0$,*

v) $\alpha(\rho) - U^*(r, \rho) = \mathcal{O}(e^{-\nu(\rho)(r-\rho)})$, $\quad r > \rho, \quad \nu(\rho) := (F''(\alpha(\rho)))^{1/2}$,
and similar exponential estimates hold for the derivative of U^* with respect to r.

A transformation to polar or spherical coordinates yields for the shifted radial component $U^\rho(s) := U^*(s + \rho, \rho)$ the ODE

$$\ddot{U}^\rho + \frac{d-1}{\rho + s}\dot{U}^\rho - F'(U^\rho) = \sigma(\rho), \quad -\rho < s < \infty. \tag{3.4}$$

With Proposition 3.4 i), we observe that $\sigma(\rho)$ tends to zero, as $\rho \to \infty$. Together with the properties ii), iii), and iv), we observe that U^ρ converges for $\rho \to \infty$ to the unique bounded solution to the heteroclinic ODE

$$\ddot{U} - F'(U) = 0, \quad U(0) = 0,\ U(\pm\infty) = \pm 1.$$

Moreover, away from the interface we can expect U^ρ to be close to one of the roots of F'. Essentially, the proof of Proposition 3.4 is a perturbation argument based on this observation. Furthermore, we see that—provided we are sufficiently far away from the center of the bubble—one can approximate the radial component U^* in terms of the heteroclinic ODE connecting the stable roots of F. For a detailed proof of this statement, we refer to Proposition 2.4 in [AF98].

Lemma 3.5 (Asymptotic expansion w.r.t. the heteroclinic, [AF98], Proposition 2.4)**.**
Let U be the unique solution to $U'' - F'(U) = 0$ subject to the boundary conditions $U(0) = 0$ and $\lim_{s\to\pm\infty} U(s) = \pm 1$. Then, there exists a constant $C > 0$ such that

i) $\sigma'(\rho) = C\rho^{-2} + \mathcal{O}(\rho^{-3})$

ii) $U^*(r, \rho) = U(r - \rho) + C\rho^{-1} V(r - \rho, \rho) + \mathcal{O}(\rho^{-2})$ *for* $r - \rho \geq -C\rho$

iii) $U^*_\rho(r, \rho) = -U'(r - \rho) + C\rho^{-2} V_\rho(r - \rho, \rho) + \mathcal{O}(\rho^{-3})$ *for* $r - \rho \geq -C\rho$,

where V is a bounded function and the subindex denotes differentiation with respect to ρ.

For a fixed radius $\rho > \bar{\rho} > 0$ of the droplets and a fixed minimal distance $\delta > 0$ from the boundary of the domain, we define for $\varepsilon \ll 1$ and $\xi \in \Omega_{\rho+\delta} := \{\xi : d(\xi, \partial\Omega) > \rho + \delta\}$ the rescaled and translated droplet state $u^\xi : \Omega \to \mathbb{R}$ by

$$u^\xi(x) = U^*\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right), \quad x \in \Omega, \tag{3.5}$$

where the number a^ξ is chosen to be zero at some fixed $\xi_0 \in \Omega_{\rho+\delta}$ and is determined for generic $\xi \in \Omega_{\rho+\delta}$ by imposing that the mass of u^ξ is constant on $\Omega_{\rho+\delta}$, i.e.,

$$\int_\Omega u^\xi \, \mathrm{d}x = \int_\Omega u^{\xi_0} \, \mathrm{d}x \qquad \forall \xi \in \Omega_{\rho+\delta}. \tag{3.6}$$

For example, we choose ξ_0 to be a point of maximal distance from the boundary $\partial\Omega$. We could also fix a small mass and then determine the radius $\rho > 0$ such that the droplet centered at ξ_0 has exactly that mass. An argument based on Proposition 3.4 shows that a^ξ and its derivatives with respect to ξ_i for $i = 1, \ldots, d$ are all exponentially small. It is only an exponentially small effect of the boundary. For more details, we refer to Lemma 3.1 in [AF98].

Lemma 3.6 ([AF98], Lemma 3.1)**.**
The number a^ξ is uniquely determined by the condition (3.6) and the assumption $a^{\xi_0} = 0$. Moreover,

$$0 \le a^\xi < Ce^{-(\nu_\varepsilon/\varepsilon)d^\xi}, \tag{3.7}$$

where $d^\xi := d(\xi, \partial\Omega) - \rho$ and $\nu_\varepsilon := \nu(\frac{\rho - a^\xi}{\varepsilon})$, with ν defined in Proposition 3.4 v). Similar exponential estimates hold for derivatives of a^ξ with respect to ξ_i, $i = 1, \ldots, d$.

Note that by Proposition 3.4 i) the root near 1 of the equation $F'(u) - \sigma(\frac{\rho - a^\xi}{\varepsilon}) = 0$ is given by $1 - \frac{\varepsilon}{\rho - a^\xi} F''(1)^{-1} + \mathcal{O}(\varepsilon^2)$. Therefore, we have

$$\nu_\varepsilon^2 = F''\left(1 - \frac{\varepsilon}{\rho - a^\xi} F''(1)^{-1} + \mathcal{O}(\varepsilon^2)\right) = F''(1) + \mathcal{O}(\varepsilon).$$

This shows that the estimate (3.7) indeed leads to an exponentially small correction term a^ξ. In the remainder of this chapter, we will denote such exponentially small terms by $\mathcal{O}(\exp)$ (cf. Definition 2.12). In fact, we do not need the exact asymptotics of the exponentially small terms, since our results are dominated by the noise strength δ_ε, which is polynomial in ε. Clearly, this is quite different to the deterministic setting. In that case, the dynamics is given by the exponentially slow motion of the droplet and hence, a more careful analysis of these terms is needed.

By virtue of Proposition 3.4, the droplet state u^ξ is an almost stationary solution to the Cahn–Hilliard equation in the sense that it fails to satisfy the equation, or the boundary conditions, by terms which are exponentially small. Moreover, it jumps from somewhere near -1 to near $+1$ in a thin layer of size of order ε around the circle of radius ρ and center ξ.

In order to fix the Neumann boundary conditions, we introduce a small perturbation v^ξ such that $\tilde{u}^\xi := u^\xi + v^\xi$ satisfies the boundary conditions $\partial_\eta \tilde{u}^\xi = \partial_\eta \Delta \tilde{u}^\xi = 0$. By virtue of Theorem 5.1 in [AF98], this can be done in such a way that $\mathcal{L}(u^\xi + v^\xi) \in \operatorname{span}\{u_i^\xi : i = 1, \ldots, d\}$. We can interpret this as the manifold consisting of the functions $u^\xi + v^\xi$ for $\xi \in \Omega_{\rho+\delta}$ being an approximate invariant manifold for the Cahn–Hilliard equation. In fact, we will collect all translates of $\tilde{u}^\xi$ in a slow manifold (Definition 3.8), but let us first state the aforementioned result in the following theorem:

Theorem 3.7 ([AF98], Theorem 5.1)**.**
Assume that $\rho > 0$ is such that $\Omega_\rho = \{\xi \in \Omega : d(\xi, \partial\Omega) > \rho\}$ is non-empty and let $\delta > 0$ be a fixed small number. Then, there is an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exist C^1-functions

$$\xi \mapsto v^\xi \in C^4(\bar{\Omega}), \qquad \xi \mapsto c^\xi = (c_1^\xi, \ldots, c_d^\xi) \in \mathbb{R}^d$$

defined in $\Omega_{\rho+\delta}$ and such that $\int_\Omega v^\xi \, \mathrm{d}x = 0$, for which

$$\text{(i) } \|v^\xi\|_\infty = \mathcal{O}(\exp) \qquad \text{and} \qquad \text{(ii) } |c^\xi| = \mathcal{O}(\exp).$$

Similar exponential estimates hold for the derivatives of v^ξ and c^ξ with respect to ξ. Moreover, the function $\tilde{u}^\xi := u^\xi + v^\xi$ satisfies the boundary conditions in (CH) and

$$\mathcal{L}(\tilde{u}^\xi) = \sum_{i=1}^d c_i^\xi u_i^\xi,$$

where $\mathcal{L}(\psi) = -\Delta(\varepsilon^2 \Delta\psi - F'(\psi))$ denotes the Cahn–Hilliard operator and u_i^ξ the derivatives of u^ξ with respect to ξ_i for $i = 1, \ldots, d$.

Motivated by Theorem 3.7, we can finally define the slow manifold $\mathcal{M}_\rho$ consisting of translates of $\tilde{u}^\xi = u^\xi + v^\xi$.

Definition 3.8 (The slow manifold)**.**
For a fixed radius $\rho > \bar{\rho} > 0$ of the droplets with $\bar{\rho}$ given by Proposition 3.4, and centers in $\Omega_{\rho+\delta} = \{\xi : d(\xi, \partial\Omega) > \rho + \delta\}$ for some fixed small $\delta > 0$, we define the *slow manifold*

$$\mathcal{M}_\rho := \left\{ \tilde{u}^\xi := u^\xi + v^\xi \, : \, \xi \in \Omega_{\rho+\delta} \right\},$$

where u^ξ denotes the droplet state defined by (3.5) and v^ξ the exponentially small perturbation defined in Theorem 3.7.

By smoothness of the functions u^ξ and v^ξ, the map $\xi \mapsto \tilde{u}^\xi$ defines a C^3-parametrization of the slow manifold $\mathcal{M}_\rho$. For $i, j \in \{1, \ldots, d\}$, we denote the partial derivatives of $\tilde{u}^\xi$ with respect to the variable ξ_j by $\tilde{u}_j^\xi := \partial_{\xi_j} \tilde{u}^\xi$, and second derivatives $\tilde{u}_{ij}^\xi := \partial_{\xi_i} \partial_{\xi_j} \tilde{u}^\xi$, accordingly. Moreover, the matrix $(\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle)_{i,j}$ is invertible (cf. Theorem 3.6 in [AFK04]). See also the proof of Lemma 3.14. Thus, $\mathcal{M}_\rho$ defines a nondegenerate d-dimensional manifold.

3.1.2 Spectral estimates for the linearized operators

An essential point (cf. Assumption 2.1) in deriving stochastic stability is the spectral properties of the linearized Cahn–Hilliard operator in H^{-1}. We consider the linearization at any droplet state in our slow manifold, and it is crucial that eigenfunctions not tangential to the manifold have negative eigenvalues uniformly bounded away from zero, while all other eigenvalues have eigenfunctions tangential to the manifold. The main spectral result is given in Theorem 3.9 and we comment on the derivation of the eigenfunctions corresponding to the exponentially small eigenvalues afterwards. Essentially, up to an exponentially small error, these eigenfunctions stem from the translation of the droplet and we will use them to approximate the tangent space of $\mathcal{M}_\rho$ and define the Fermi coordinates.

Besides, we will see that the spectrum of the linearized mass conserving Allen–Cahn operator plays an important role in the analysis. Here, we give the main spectral result in Theorem 3.12.

The Cahn–Hilliard operator on $H_0^{-1}(\Omega)$

For $\tilde{u}^\xi \in \mathcal{M}_\rho$, we study the linearized Cahn–Hilliard operator

$$\mathcal{L}^\xi = -\Delta\left(\varepsilon^2\Delta - F''(\tilde{u}^\xi)\right)$$

as an operator on $H_0^{-1}(\Omega)$ in more detail. The droplet is stable for the dynamics and hence, the exponentially small eigenvalues of $\mathcal{L}^\xi$ stem (up to an exponentially small error) from translations of the droplet. Crucial for the stability analysis (see Section 3.3) is the spectral gap, which as we will see depends on the space dimension d. For its consequences on the stability results see Remark 3.10. The spectrum of $\mathcal{L}^\xi$ was analyzed in [AF94] and we cite the full result below.

Theorem 3.9 (The linearized Cahn–Hilliard operator, [AF94])**.**
Let $d \in \{2,3\}$ and $\tilde{u}^\xi \in \mathcal{M}_\rho$.

(i) The operator $\mathcal{L}^\xi$ can be extended to a self-adjoint operator on H_0^{-1}. Moreover, $-\mathcal{L}^\xi$ is bounded from below.

(ii) Let $\lambda_1^\xi \le \lambda_2^\xi \le \lambda_3^\xi \le \dots$ be the eigenvalues of

$$\begin{cases} \mathcal{L}^\xi\psi = -\Delta\left(\varepsilon^2\Delta\psi - F''(\tilde{u}^\xi)\psi\right) = -\lambda\psi, & x \in \Omega, \\ \dfrac{\partial\psi}{\partial\eta} = \dfrac{\partial\Delta\psi}{\partial\eta} = 0, & x \in \partial\Omega. \end{cases}$$

Then, there exist $\varepsilon_0 > 0$ and a constant $C > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the following estimates hold true:

$$|\lambda_d^\xi| = \mathcal{O}(\exp) \quad \textit{and} \quad \lambda_{d+1}^\xi \ge C\varepsilon^{d-1}.$$

(iii) In the d-dimensional subspace U^ξ corresponding to the exponentially small eigenvalues $\lambda_1^\xi, \dots, \lambda_d^\xi$, there is an H^{-1}-orthonormal basis $\psi_1^\xi, \dots, \psi_d^\xi$ such that

$$\psi_i^\xi = \sum_{j=1}^{d} a_{ij}^\xi \frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|} + \mathcal{O}(\exp), \quad i = 1, \dots, d, \tag{3.8}$$

where the matrix (a_{ij}^ξ) is non-singular and a smooth function of ξ. Moreover, ψ_i^ξ is a smooth function of ξ and

$$\|\psi_{i,j}^\xi\| = \mathcal{O}(\varepsilon^{-1}), \quad i, j = 1, \dots, d, \tag{3.9}$$

where $\psi_{i,j}^\xi$ denotes the derivative of ψ_i^ξ with respect to ξ_j.

Remark 3.10 (Dependence on the space dimension)**.**
Thus far, the spectral gap in Theorem 3.9(ii) depends decisively on the space dimension d. This heavily influences our analysis of stochastic stability, and any improvement in this result will yield a better region of stability in the three-dimensional setting. Basically, the smaller spectral gap will weaken the estimate of Metatheorem 2 and, due to the Sobolev embeddings used for the proof, reduce the maximal radius R_ε of the tubular neighborhood around the slow manifold that we can treat. This directly influences the main stability result of Theorem 2.14 and will only allow for a smaller noise strength. In fact, we have seen the dependence of the noise strength on the space dimension d already in Assumption 3.2, where we assumed that $\delta_\varepsilon < \varepsilon^{7/2+3d/2}$. For the exact interplay between the spectral gap and stochastic stability, we refer to Section 3.3.

As we will need the statement in more detail later, we comment briefly on the proof of (iii). The main ingredient is the following theorem.

Theorem 3.11 ([HS84]).
Let $\mathcal{A}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$, I a compact interval in $\mathbb{R}$, and $\{\psi_1, \dots, \psi_N\}$ linearly independent normalized elements in $\mathcal{D}(\mathcal{A})$. Additionally, suppose that

(i) For some $\varepsilon' > 0$ and $\mu_j \in I$, $j = 1, \dots, N$, we have

$$\mathcal{A}\psi_j \;=\; \mu_j \psi_j + r_j \quad \textit{with} \quad \|r_j\| \;<\; \varepsilon'.$$

(ii) There is a number $a > 0$ such that I is a-isolated in the spectrum of $\mathcal{A}$, i.e.,

$$(\sigma(A) \setminus I) \,\cap\, (I + (-a, a)) \;=\; \emptyset.$$

Then, we obtain

$$\bar{d}(E, F) \;:=\; \sup_{\varphi \in E, \|\varphi\|=1} d(\varphi, F) \;\leq\; \frac{\sqrt{N}\varepsilon'}{a\sqrt{\lambda_{min}}},$$

where

$$\begin{aligned} E \;&=\; \mathrm{span}\{\psi_1, \dots, \psi_N\}, \\ F \;&=\; \textit{closed subspace associated to the eigenvalues in } \sigma(A) \cap I, \\ \lambda_{min} \;&=\; \textit{smallest eigenvalue of the matrix } (\langle \psi_i, \psi_j \rangle)_{i,j=1,\dots,N}\,. \end{aligned}$$

For the application to the linearized Cahn–Hilliard operator, we set

$$E \;:=\; \mathrm{span}\left\{ \frac{\tilde{u}_1^\xi}{\|\tilde{u}_1^\xi\|}, \dots, \frac{\tilde{u}_d^\xi}{\|\tilde{u}_d^\xi\|} \right\} \quad \text{and} \quad I \;:=\; \left[-Ce^{-c/\varepsilon}, Ce^{-c/\varepsilon}\right],$$

for some constants $c, C > 0$ fitting with the estimate of the exponentially small eigenvalues in Theorem 3.9(ii). Also, note that the functions $\tilde{u}_i^\xi$ are linearly independent, since the matrix $\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle_{i,j=1,\dots,d}$ is invertible (cf. [AFK04], Theorem 3.6).
Obviously, we have $\sigma(\mathcal{A}) \cap I = \{\lambda_1^\xi, \dots, \lambda_d^\xi\}$ and, as according to Theorem 3.9(ii) the spectral gap is of order ε^{d-1}, the interval I is ε^d-isolated in the spectrum of the linearized Cahn–Hilliard operator. Furthermore, we have by Theorem 3.7

$$\mathcal{L}^\xi \frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|} \;=\; \frac{1}{\|\tilde{u}_j^\xi\|} \left(\sum_{i=1}^d \frac{\partial c_j^\xi}{\partial \xi_i} u_j^\xi + c_j^\xi u_{ij}^\xi \right) \;=\; \mathcal{O}(\exp).$$

Since the matrix $(\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle)$ approaches a non-singular limit as ε tends to zero, we also obtain that the smallest eigenvalue is uniformly bounded away from zero. For $i \in \{1, \dots, d\}$, let ψ_i^ξ be the eigenvector associated to the eigenvalue λ_i^ξ and define $F := \mathrm{span}\{\psi_1^\xi, \dots, \psi_d^\xi\}$. With that, Theorem 3.11 is finally applicable and yields

$$\bar{d}(E, F) \;:=\; \sup_{\varphi \in E, \|\varphi\|=1} d(\varphi, F) \;=\; \mathcal{O}(\exp).$$

Hence, by definition of the distance $\bar{d}$, we have for some $\alpha \in \mathbb{R}^d$

$$\frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|} \;=\; \sum_{k=1}^d \alpha_k \, \psi_k^\xi + \mathcal{O}(\exp), \quad j \;=\; 1, \dots, d.$$

By taking the inner product with the orthonormal basis ψ_i^ξ, we can solve for the coefficients α_k and obtain directly that

$$\frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|} = \sum_{k=1}^{d} \langle \psi_k^\xi, \frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|}\rangle \, \psi_k^\xi + \mathcal{O}(\exp). \tag{3.10}$$

Provided that $\varepsilon > 0$ and the radius ρ of the droplet are sufficiently small, the matrix $B(\xi)$ defined by $B_{ij}(\xi) = \langle \psi_i^\xi, \tilde{u}_j^\xi \rangle$ is invertible. In more detail, relation (3.10) together with the orthonormal basis $\{\psi_i^\xi\}_{i=1,\dots,d}$ implies (ignoring exponentially small terms)

$$\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle = \Big\langle \sum_{k=1}^{d} B_{ki}(\xi)\, \psi_k^\xi, \sum_{\ell=1}^{d} B_{\ell j}(\xi)\, \psi_\ell^\xi \Big\rangle = \sum_{k,\ell=1}^{d} B_{ki}(\xi) B_{\ell j}(\xi) \langle \psi_k^\xi, \psi_\ell^\xi \rangle = \Big(B^T \cdot B\Big)_{ij}. \tag{3.11}$$

Therefore, invertibility of B is equivalent to the invertibility of the matrix defined by $\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle$, which is crucial for the non-degeneracy of the slow manifold $\mathcal{M}_\rho$ (see also Lemma 3.14).
Thus, we have $\psi_i^\xi \in E + \mathcal{O}(\exp)$ as

$$\psi_i^\xi = \sum_{j=1}^{d} \|\tilde{u}_j^\xi\| B_{ij}^{-T}(\xi) \frac{\tilde{u}_j^\xi}{\|\tilde{u}_j^\xi\|} + \mathcal{O}(\exp), \quad i = 1, \dots, d. \tag{3.12}$$

The mass conserving Allen–Cahn operator on $L_0^2(\Omega)$

By defining the projection $\mathrm{P}u := u - |\Omega|^{-1} \int_\Omega u \,\mathrm{d}x$ onto $L_0^2(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f(x)\,\mathrm{d}x = 0\}$, we obtain for $v \in H_0^{-1}$

$$\begin{aligned}\langle \mathcal{L}^\xi v, v \rangle_{H_0^{-1}} &= \langle \varepsilon^2 \Delta v - F''(\tilde{u}^\xi) v, \mathrm{P}v \rangle_{L^2} = \langle \mathrm{P}[\varepsilon^2 \Delta v - F''(\tilde{u}^\xi) v], v \rangle_{L^2} \\ &= \langle \varepsilon^2 \Delta v - F''(\tilde{u}^\xi) v + \frac{1}{|\Omega|} \int_\Omega F''(\tilde{u}^\xi) v \,\mathrm{d}x, v \rangle_{L^2} := \langle \mathcal{A}^\xi v, v \rangle_{L^2}.\end{aligned}$$

Therefore, it is fruitful to study the eigenvalue problem for the mass conserving Allen–Cahn equation on $L^2(\Omega)$ linearized at the droplet state $\tilde{u}^\xi \in \mathcal{M}_\rho$

$$\begin{cases} \mathcal{A}^\xi \psi = \varepsilon^2 \Delta \psi - F''(\tilde{u}^\xi) \psi + \dfrac{1}{|\Omega|} \displaystyle\int_\Omega F''(\tilde{u}^\xi) \psi = -\mu \psi, & x \in \Omega, \\ \partial_\eta \psi = 0, & x \in \partial\Omega. \end{cases} \tag{3.13}$$

The main spectral result concerning the eigenvalue problem (3.13) is the following theorem. This result can be found in [ABF98], with $\tilde{u}^\xi$ replaced by u^ξ. Since the difference $\tilde{u}^\xi - u^\xi$ is exponentially small, the theorem follows from an easy perturbation argument.

Theorem 3.12 (The linearized Allen–Cahn operator, [ABF98], Proposition 2.2)**.**
Let $\tilde{u}^\xi \in \mathcal{M}_\rho$ and $\mu_1^\xi \leq \mu_2^\xi \leq \dots$ be the eigenvalues of (3.13). Then, there exist $\varepsilon_0 > 0$ and a constant $C > 0$ such that for $0 < \varepsilon < \varepsilon_0$

$$\mu_1^\xi, \dots, \mu_d^\xi = \mathcal{O}(\exp) \quad \textit{and} \quad \mu_{d+1}^\xi > C\varepsilon^2.$$

The d-dimensional space W^ξ spanned by the eigenfunctions corresponding to the exponentially small eigenvalues $\mu_1^\xi, \dots, \mu_d^\xi$ can be represented by $W^\xi = \mathrm{span}\{w_1^\xi, \dots, w_d^\xi\}$, where the normalized eigenfunctions w_i^ξ are estimated by

$$\Big\| w_i^\xi - \frac{\tilde{u}_i^\xi}{\|\tilde{u}_i^\xi\|} \Big\| = \mathcal{O}(\exp).$$

3.2 Motion along the slow manifold

In this section, we follow the general approach presented in Section 2.2 and give the effective equation on the slow manifold $\mathcal{M}_\rho$. Afterwards, we analyze the ODE governing the motion of the droplet's center in terms of ε being small. We show that the droplet moves in first approximation according to the projection of the Wiener process onto the slow manifold (cf. Section 2.2.3). Moreover, since the dominating terms for the dynamics are not small in ε but in the inverse radius ρ^{-1}, we expect that the size of the bubble influences the speed of motion, i.e., smaller droplets move faster.

3.2.1 The new coordinate system

In order to derive the effective dynamics, we first have to introduce a coordinate system in a small tubular neighborhood of the slow manifold $\mathcal{M}_\rho$ (Fermi coordinates, see Definition 2.2). A minor technical difficulty is that the eigenfunctions $\psi_1^\xi, \ldots, \psi_d^\xi$ of the linearization $\mathcal{L}^\xi$ do not span the tangent space at a given point $\tilde{u}^\xi$ on the slow manifold. But, as the difference to the true tangent space spanned by the partial derivatives $\tilde{u}_1^\xi, \ldots, \tilde{u}_d^\xi$ is exponentially small, we can use them as an approximate tangent space to project onto the manifold (cf. Remark 2.3). The following proposition (see [AF98], Proposition 7.1) concerns the existence of a small tubular neighborhood in H^{-1} of radius $\mathcal{O}(\varepsilon^{1+})$ around $\mathcal{M}_\rho$, where the projection is well defined.
We do not give a proof, as we do not need the statement in this generality. The uniqueness of the Fermi coordinates can also be inferred from the local existence of the system governing the motion of the droplet's center ξ (cf. Section 2.2.2).

Proposition 3.13 (Fermi coordinates, [AF98], Proposition 7.1)**.**
Let $\tilde{u}^\xi$, $\mathcal{M}_\rho$, and Ω_ρ be as in Theorem 3.7. Then, for $\eta > 1$, the condition

$$\inf_{\xi \in \Omega_{\rho+2\delta}} \|u - \tilde{u}^\xi\| < \varepsilon^\eta, \tag{3.14}$$

implies the existence of unique $\xi \in \Omega_{\rho+\delta}$ and $v \in H_0^{-1}$ such that

$$u = \tilde{u}^\xi + v, \quad \langle v, \psi_i^\xi \rangle = 0 \quad \forall i = 1, \ldots, d, \tag{3.15}$$

where the eigenfunctions $\psi_1^\xi, \ldots, \psi_d^\xi$ are given by Theorem 3.9 (iii). Moreover, the map $u \mapsto (\xi, v)$ defined by (3.15) is smooth together with its inverse.

Let $u(t)$ be a solution of (CH). We call the coordinates $(\xi(t), v(t))$ defined in Proposition 3.13 the *Fermi coordinates* of $u(t)$ (cf. Definition 2.2).

3.2.2 The exact stochastic equation for the droplet's motion

With the new coordinate frame at hand, we are now equipped with all necessary tools to give the exact stochastic equation on the slow manifold $\mathcal{M}_\rho$. Following the guideline from Chapter 2, we first have to show that the matrix $A(\xi, v) \in \mathbb{R}^{d\times d}$ from Definition & Metatheorem 1 is invertible, as long as $\|v\|$ stays sufficiently small. Then, the effective dynamics was established in Theorem 2.6 and we can rely on these formulas for the drift and diffusion. Note that in the computation of the effective dynamics we rely on an approximation of the tangent space $\mathcal{T}_{\tilde{u}^\xi}\mathcal{M}_\rho$ by the exact eigenfunctions ψ_i^ξ (cf. Remark 2.3). While this is not crucial for the stochastic ODE and its approximation in terms of ε, it helps with the stability analysis in Section 3.3, as we can apply the spectral estimates directly.

Lemma 3.14 (Invertibility of the matrix A).
For $\tilde{u}^\xi \in \mathcal{M}_\rho$ and ψ_i^ξ, $i = 1, \ldots, d$, given by Theorem 3.9(iii), consider the matrix $A(\xi, v) \in \mathbb{R}^{d\times d}$ defined by

$$A_{kj}(\xi, v) \;=\; B_{kj} - R_{kj}(v) \;\coloneqq\; \langle \psi_k^\xi, \tilde{u}_j^\xi \rangle - \langle v, \psi_{k,j}^\xi \rangle.$$

Then, as long as $\|v\| \le C\varepsilon^{1+\kappa}$ for some $0 < \varepsilon < \varepsilon_0$ and some small $\kappa > 0$, the matrix $A(\xi, v)$ is invertible. Moreover, for $\rho < 1$ and some $C_0 > 0$, its inverse $A^{-1}(\xi, v)$ is given by

$$A_{kj}^{-1}(\xi, v) \;=\; C_0^{-1}\rho^{-d/2}\,\mathrm{I}_d + \mathcal{O}(\rho^{d/2-1}).$$

Proof. By [AFK04, Theorem 3.6], we have for a specific constant $C_0 > 0$ that

$$\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle \;=\; C_0^2\rho^d\,\delta_{ij} + \mathcal{O}(\rho^{2d-1}) + \mathcal{O}(\varepsilon\rho^{-1}) + \mathcal{O}(\exp). \tag{3.16}$$

Therefore, $\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle$ defines for small ρ an almost diagonal, invertible matrix of order $\mathcal{O}(1)$ in ε. Moreover, we see that $\|\tilde{u}_i^\xi\|^2 = C_0^2\rho^d + \mathcal{O}(\rho^{2d-1})$. In (3.11), we proved the relation

$$\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle \;=\; \Big(B^T \cdot B\Big)_{ij} + \mathcal{O}(\exp), \tag{3.17}$$

such that invertibility of B can be derived from the invertibility of the matrix $(\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi \rangle)_{i,j}$. On the other hand, the assumption on $\|v\|$ and $\|\psi_{i,j}\| = \mathcal{O}(\varepsilon^{-1})$ yields

$$\langle v, \psi_{i,j}^\xi \rangle \;\le\; C\|v\|\|\psi_{i,j}^\xi\| \;\le\; C\varepsilon^\kappa.$$

From this, we see directly that $A(\xi, v)$ is invertible for ε sufficiently small.
Using the relations (3.16) and (3.17), we obtain $B = C_0\rho^{d/2}\,\mathrm{I}_d + \mathcal{O}(\rho^{3d/2-1})$. Let us now consider the decomposition $A(\xi, v) = C_0\rho^{d/2}(\mathrm{I}_d - E)$, where I_d denotes the identity matrix and E is a small perturbation thereof of order $\mathcal{O}(\rho^{d-1})$. Then, one has by Taylor expansion (or geometric series)

$$\begin{aligned} A(\xi, v)^{-1} \;&=\; C_0^{-1}\rho^{-d/2}(\mathrm{I}_d - E)^{-1} \;=\; C_0^{-1}\rho^{-d/2}\sum_{k\in\mathbb{N}} E^k \\ &=\; C_0^{-1}\rho^{-d/2}(\mathrm{I}_d + E + \mathcal{O}(\rho^{2d-2})) \;=\; C_0^{-1}\rho^{-d/2}\,\mathrm{I}_d + \mathcal{O}(\rho^{d/2-1}). \end{aligned}$$

With this estimate, the lemma is proved. □

Provided the matrix A is invertible, we gave the equation of the full dynamics in Theorem 2.6. Under the assumption that ξ performs a d-dimensional diffusion process

$$d\xi \;=\; b(\xi, v)\,dt + \langle \sigma(\xi, v), dW \rangle, \tag{3.18}$$

the drift term $b : \mathbb{R}^d \times H^{-1}(\Omega) \to \mathbb{R}^d$ and the diffusion $\sigma : \mathbb{R}^d \times H^{-1}(\Omega) \to (H^{-1}(\Omega))^d$ are then given by the expressions

$$\sigma_r(\xi, v) \;=\; \sum_{i=1}^d A_{ri}^{-1}(\xi)\psi_i^\xi \tag{3.19}$$

and

$$\begin{aligned} b_r(\xi, v) \;=& \sum_{i=1}^d A_{ri}^{-1}(\xi)\langle \mathcal{L}(v + \tilde{u}^\xi), \psi_i^\xi \rangle + \sum_{i=1}^d A_{ri}^{-1}(\xi)\sum_j \langle \mathcal{Q}\psi_{i,j}^\xi, \sigma_j^\xi \rangle \\ &+ \sum_{i,j,k=1}^d A_{ri}^{-1}(\xi)\left[\tfrac{1}{2}\langle v, \psi_{i,jk}^\xi \rangle - \langle \psi_{j,k}^\xi, \tilde{u}_i^\xi \rangle - \tfrac{1}{2}\langle \psi_k^\xi, \tilde{u}_{ij}^\xi \rangle\right]\langle \mathcal{Q}\sigma_i^\xi, \sigma_j^\xi \rangle. \end{aligned} \tag{3.20}$$

Note that we justified this ansatz in Lemma 2.8.

3.2.3 Bounds on the SDE

In the present section, we give bounds on the SDE governing the motion of the droplet's center ξ, which we need later in Section 3.3 to show stochastic stability. Recall that in Assumption 3.2 we defined a weighted H^1-norm $\|\cdot\|_{H^1_\varepsilon}$ by

$$\|v\|_{H^1_\varepsilon} = \left(\int_\Omega \varepsilon^2|\nabla v|^2 + v^2\,\mathrm{d}x\right)^{1/2}. \tag{3.21}$$

Obviously, this norm is equivalent to the usual H^1-norm with ε-dependent constants and we always have that

$$\|v\|_{L^2} \le \|v\|_{H^1_\varepsilon} \qquad \text{and} \qquad \|\nabla v\|_{L^2} \le \varepsilon^{-1}\|v\|_{H^1_\varepsilon}.$$

Moreover, by the Poincaré inequality we obtain for the H^{-1}-norm

$$\|v\|_{H^{-1}} \le \|v\|_{L^2} \le \|v\|_{H^1_\varepsilon}.$$

For the purpose of defining tubular coordinates (Proposition 3.13) and invertibility of the matrix $A(\xi,v)$ (Lemma 3.14), we needed an H^{-1}-radius of order $\mathcal{O}(\varepsilon^{1+\kappa})$ for some $\kappa>0$. Since it is not sufficient to control the nonlinear terms in the drift term b only by means of the H^{-1}-norm, we work in a small tubular neighborhood of $\mathcal{M}_\rho$ defined by $\|v\|_{H^1_\varepsilon}$ being sufficiently small. By virtue of the above mentioned Poincaré inequality, an H^1_ε-radius of order $\mathcal{O}(\varepsilon^{1+\kappa})$ suffices to have the coordinate frame, and thus the SDE, well-defined.

Definition 3.15. For some small $\kappa>0$, we define a tubular neighborhood of the slow manifold $\mathcal{M}_\rho$ by

$$\Gamma := \left\{\tilde{u}^\xi + v \,:\, \xi\in\Omega_{\rho+\delta},\ \|v\|_{H^1_\varepsilon} < \varepsilon^{1+\kappa}\right\}.$$

As long as $\tilde{u}^\xi+v$ lies in Γ—i.e., as long as the coordinate system (3.15) is well-defined—we give bounds on the stochastic ODE. Later, when we analyze the SDE in more detail, we will use an even smaller tubular neighborhood, where solutions stay inside for a very long time. Let us start with estimating the diffusion term $\sigma(\xi,v)$.

Lemma 3.16. *Let $\tilde{u}^\xi+v\in\Gamma$ and $r\in\{1,\dots,d\}$. We obtain*

$$\sigma_r(\xi,v) = C_0^{-1}\rho^{-d/2}\psi_r^\xi + \mathcal{O}_{H^{-1}}(\rho^{d/2-1}),$$

where $C_0>0$ is the constant from Lemma 3.14.

Proof. By (3.19), we have

$$\sigma_r(\xi,v) = \sum_{i=1}^d A_{ri}^{-1}(\xi)\psi_i^\xi.$$

The claim follows directly from the estimate of the inverse A^{-1} in Lemma 3.14, and the eigenfunctions ψ_i^ξ being normalized in $H_0^{-1}(\Omega)$. □

To complete the estimates on the stochastic ODE, it remains to bound the drift term $b(\xi,v)$ defined by (3.20). Note that only here we need to assume smallness in H^1_ε in order to handle the nonlinear terms.

Lemma 3.17. *Let $\tilde{u}^\xi+v\in\Gamma$. We obtain*

$$|b(\xi,v)| = \mathcal{O}(\varepsilon^{2-d/2+3\kappa} + \varepsilon^{-1}\delta_\varepsilon^2).$$

Proof. Via Taylor expansion we obtain

$$\langle \mathcal{L}(v+\tilde{u}^\xi), \psi_i^\xi \rangle = \langle \mathcal{L}(\tilde{u}^\xi), \psi_i^\xi \rangle + \langle \mathcal{L}^\xi v, \psi_i^\xi \rangle + \langle \mathcal{N}^\xi(v), \psi_i^\xi \rangle,$$

where $\mathcal{N}^\xi(v)$ collects the remaining nonlinear terms and is given by $\mathcal{N}^\xi(v) = \Delta(3\tilde{u}^\xi v^2 + v^3)$. By Theorem 3.9, we have $\|\mathcal{L}(\tilde{u}^\xi)\|_\infty = \mathcal{O}(\exp)$ and $\langle \mathcal{L}^\xi v, \psi_i^\xi \rangle = 0$, as $\mathcal{L}^\xi v \perp \psi_i^\xi$ due to invariance of the operator $\mathcal{L}^\xi$. Using that $\tilde{u}^\xi$ is uniformly bounded and $\|\psi_i^\xi\|_\infty = \mathcal{O}(\varepsilon^{-1})$ (see the estimates in Lemma 3.33), Nirenberg's inequality implies for the nonlinear terms

$$\begin{aligned} |\langle \mathcal{N}^\xi(v), \psi_i^\xi \rangle| &= \Big| \int_\Omega (3\tilde{u}^\xi v^2 + v^3)\psi_i^\xi \, \mathrm{d}x \Big| \\ &\leq C\varepsilon^{-1} \Big[\|v\|_{L^2}^2 + \|v\|_{L^3}^3 \Big] \\ &\leq C\varepsilon^{-1} \Big[\|v\|_{H_\varepsilon^1}^2 + \|v\|_{L^2}^{3-d/2} \|\nabla v\|_{L^2}^{d/2} \Big] \\ &\leq C\varepsilon^{-1} \Big[\|v\|_{H_\varepsilon^1}^2 + \varepsilon^{-d/2} \|v\|_{H_\varepsilon^1}^3 \Big] \leq C\varepsilon^{2-d/2+3\kappa}, \end{aligned} \tag{3.22}$$

where we interpolated the L^3-norm between L^2 and H^1.
As a next step, we analyze the terms in (3.20) that appear due to Itô calculus and thus depend on the noise strength δ_ε. We utilize that $\|\psi_{i,j}^\xi\| = \mathcal{O}(\varepsilon^{-1})$ by (3.9) and the bound on σ from Lemma 3.16, i.e., $\|\sigma\| = \mathcal{O}(1)$ in ε. This yields via the Cauchy-Schwarz inequality

$$|\langle \mathcal{Q}\psi_{i,j}^\xi, \sigma_j \rangle| \leq \|\mathcal{Q}\|_{L(H^{-1})} \|\psi_{i,j}^\xi\|_{H^{-1}} \|\sigma_j\|_{H^{-1}} \leq c\delta_\varepsilon^2 \varepsilon^{-1}.$$

For the remaining terms, we use that $\|\psi_{i,jk}^\xi\| = \mathcal{O}(\varepsilon^{-3/2})$, $\|\tilde{u}_i^\xi\| = \mathcal{O}(1)$, and $\|\tilde{u}_{ij}^\xi\| = \mathcal{O}(\varepsilon^{-1/2})$. Heuristically, the H^{-1}-norm eliminates one derivative and, for example, the H^{-1}-norm of second derivatives of $\tilde{u}^\xi$ behaves like first derivatives in L^2. Since $\nabla_\xi \tilde{u}^\xi$ is $\mathcal{O}(\varepsilon^{-1})$ on a set of order ε, one then obtains that $\|\tilde{u}_{ij}^\xi\| = \mathcal{O}(\varepsilon^{-1/2})$. For the detailed proof we refer to Lemma 3.34. Using these estimates yields

$$|\langle v, \psi_{i,jk}^\xi \rangle| \leq \|v\|_{H_\varepsilon^1} \|\psi_{i,jk}^\xi\| \leq c\varepsilon^{-1/2+\kappa}, \quad |\langle \psi_{j,k}^\xi, \tilde{u}_i^\xi \rangle| \leq c\varepsilon^{-1}, \quad \text{and} \quad |\langle \psi_k^\xi, \tilde{u}_{ij}^\xi \rangle| \leq c\varepsilon^{-1/2}.$$

Note that by Lemma 3.16 and Assumption 3.2 on the noise strength $\langle \mathcal{Q}\sigma_i, \sigma_j \rangle \leq C\delta_\varepsilon^2$. By Lemma 3.14, the matrix A^{-1} is bounded by a constant and thus the claim is verified. □

3.2.4 Approximate stochastic ODE for the droplet's motion

We investigate the stochastic ODE for the droplet's motion in more detail. We show that the dynamics of the center ξ is in first approximation given by the projection of the Wiener process onto $\mathcal{M}_\rho$ (Theorem 3.19). Moreover, we will see that smaller droplets move faster than larger ones (Remark 3.21). We observe that, under the assumptions of Lemma 3.17 and the postulated noise strength $\delta_\varepsilon < \varepsilon^{7/2+3d/2}$, the deterministic term $\sum \mathcal{A}_{ri}^{-1} \langle \mathcal{L}(\tilde{u}^\xi + v), \psi_i^\xi \rangle$ is dominating the dynamics. Therefore, we will adept the H_ε^1-radius for the analysis of the SDE. For this purpose, we define a neighborhood of the manifold $\mathcal{M}_\rho$ by

$$\Gamma' := \Big\{ \tilde{u}^\xi + v \; : \; \xi \in \Omega_{\rho+\delta}, \; \|v\|_{H_\varepsilon^1} < \varepsilon^{-1/2-d/2}\delta_\varepsilon \Big\}. \tag{3.23}$$

Note that by Assumption 3.2 we have $\Gamma' \subset \Gamma$ and thus the coordinate system (3.15) is well-defined in Γ'. While Γ is the set where the SDE is well-defined, Γ' is a smaller neighborhood of $\mathcal{M}_\rho$ from which with high probability solutions do not exit for long times unless the droplet reaches the boundary of Ω (see Section 3.3). First, let us proof that (up to a small error) the motion of the droplet's center ξ is given by the projection of the Wiener process onto the tangent space of $\mathcal{M}_\rho$ at $\tilde{u}^\xi$ (cf. Section 2.2.3).

Lemma 3.18. *As long as $\tilde u^\xi + v \in \Gamma'$, we have*

$$d\xi_r = \sum_{i=1}^d S_{ri}^{-1}\langle \tilde u_i^\xi, \circ\, dW\rangle + \mathcal{O}_{H^{-1}}(\varepsilon^{-2-d}\delta_\varepsilon^2)\, dt + \langle \mathcal{O}_{H^{-1}}(\varepsilon^{-3/2-d/2}\delta_\varepsilon), \circ\, dW\rangle,$$

where the matrix S is given by $S_{ij} = \langle \tilde u_i^\xi, \tilde u_j^\xi\rangle$.

Proof. By redoing the computation of Section 2.2 that led to the explicit formula for the effective dynamics in Theorem 2.6 in the Stratonovich sense, and thereby leaving out Itô correction terms, we obtain

$$d\xi_r = \sum_{i=1}^d A_{ri}^{-1}\langle \mathcal{L}(\tilde u^\xi + v), \psi_i^\xi\rangle\, dt + \sum_{i=1}^d A_{ri}^{-1}\langle \psi_i^\xi, \circ\, dW\rangle.$$

By using the bound (3.22) from the proof of Lemma 3.17, we immediately derive

$$\sum_{i=1}^d A_{ri}^{-1}\langle \mathcal{L}(\tilde u^\xi + v), \psi_i^\xi\rangle \le C\varepsilon^{-1}\Big[\|v\|_{H_\varepsilon^1}^2 + \varepsilon^{-d/2}\|v\|_{H_\varepsilon^1}^3\Big] \le C\varepsilon^{-2-d}\delta_\varepsilon^2,$$

and hence

$$d\xi_r = \mathcal{O}_{H^{-1}}(\varepsilon^{-2-d}\delta_\varepsilon^2)\, dt + \sum_{i=1}^d A_{ri}^{-1}\langle \psi_i^\xi, \circ\, dW\rangle.$$

For shorthand notation, we define the vectors $\psi^\xi := (\psi_1^\xi, \ldots, \psi_d^\xi)$ and $\partial_\xi \tilde u^\xi := (\tilde u_1^\xi, \ldots, \tilde u_d^\xi)$. With that definition, the matrix S is then given by $\langle \partial_\xi \tilde u^\xi, \partial_\xi \tilde u^\xi\rangle$. Moreover, using the notation of Lemma 3.14, we denote the matrix consisting of all inner products of ψ^ξ and $\partial_\xi \tilde u^\xi$ by $B = \langle \psi^\xi, \partial_\xi \tilde u^\xi\rangle$, and the small perturbation $R(v)$ thereof by $R(v) = \langle \partial_\xi \psi^\xi, v\rangle$. By definition, we have $A = B - R(v)$. Also, note that in Γ' we have $|R(v)| = \mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon)$. With Lemma 3.14 we derive

$$A^{-1} = B^{-1} + \mathcal{O}\Big(|B^{-1}R(v)B^{-1}|\Big) = B^{-1} + \mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon).$$

Moreover, the relations (3.11) and (3.12) together imply

$$B^{-1}\psi^\xi = B^{-1}B^{-T}\partial_\xi \tilde u^\xi + \mathcal{O}(\exp) = (B^T B)^{-1}\partial_\xi \tilde u^\xi + \mathcal{O}(\exp) = S^{-1}\partial_\xi \tilde u^\xi + \mathcal{O}(\exp). \qquad \square$$

We use Lemma 3.18 to show that the expected distance between the exact solution $\xi(t)$ and the projection $h(t)$ of the Wiener process W onto $\mathcal{M}_\rho$ stays small up to times of order $\mathcal{O}(\varepsilon^{2+d+\ell}\delta_\varepsilon^{-2})$ for some $\ell > 0$. After we formulated and proved the approximation result, we will see that we have to choose $\ell > 2 + d$ to obtain a reasonable error estimate. This time scale does not quite correspond to the times when the droplet reaches the boundary. Basically, any improvement of the stability region increases the time scales we can treat. Also, note that the projection $h(t)$ of the Wiener process onto $\mathcal{M}_\rho$ is given by the Stratonovich SDE (cf. Section 2.2.3)

$$dh_r = \sum_{i=1}^d S_{ri}(h)^{-1}\langle \tilde u_i^h, \circ\, dW\rangle, \quad \text{where} \quad S_{ri}(h) = \langle \tilde u_r^h, \tilde u_i^h\rangle. \tag{3.24}$$

Theorem 3.19. *Let $\xi(t)$ be the solution to (3.18) and $h(t)$ the projection of the Wiener process W onto the tangent space $\mathcal{T}_{\tilde u^{h(t)}}\mathcal{M}_\rho$ given by (3.24). Then, for any $\ell > 0$ there exists a constant $C > 0$ such that*

$$\mathbb{E} \sup_{0\le t<\varepsilon^{2+d+\ell}\delta_\varepsilon^{-2}\wedge\tau} |\xi(t) - h(t)| \le C\varepsilon^\ell + C\varepsilon^{-1/2+\ell/2}\delta_\varepsilon,$$

where τ denotes the first exit time from Γ'.

To prove the approximation result, we first have to show that the map $h \mapsto \sum_i S_{ri}(h)^{-1}\tilde{u}_i^h$ is Lipschitz continuous. Also, we compute the Lipschitz constant explicitly.

Lemma 3.20. *Let $h, \bar{h} \in \Omega_{\rho+\delta}$. Then, we have for any $r \in \{1, \dots, d\}$*

$$\sum_{i=1}^{d} \|S_{ri}(h)^{-1}\tilde{u}_i^h - S_{ri}(\bar{h})^{-1}\tilde{u}_i^{\bar{h}}\| \;\leq\; c\varepsilon^{-1/2}|h-\bar{h}|.$$

Proof. We compute $\partial_{h_k} S_{ri}(h) = \langle \tilde{u}_{rk}^h, \tilde{u}_i^h\rangle + \langle \tilde{u}_r^h, \tilde{u}_{ik}^h\rangle$ and thus, using the estimates $\|\tilde{u}_i^\xi\| = \mathcal{O}(1)$ and $\|\tilde{u}_{rk}^\xi\| = \mathcal{O}(\varepsilon^{-1/2})$ from Lemma 3.34, we obtain $|D_h S(h)| = \mathcal{O}(\varepsilon^{-1/2})$. This shows that the derivative of the inverse can be estimated by

$$|D_h S^{-1}(h)| \;=\; |S^{-1}(D_h S)S^{-1}| \;=\; \mathcal{O}(\varepsilon^{-1/2}),$$

where we utilized that $S^{-1} = C\rho^{-d}\left[I + \mathcal{O}(\rho^{d-1})\right]$ by (3.16). Therefore, with the estimates from Lemma 3.34 we obtain

$$\frac{\partial}{\partial h_k}\left(S_{ri}(h)^{-1}\tilde{u}_i^h\right) \;=\; \frac{\partial S_{ri}(h)^{-1}}{\partial h_k}\tilde{u}_i^h + S_{ri}(h)^{-1}\tilde{u}_{ik}^h \;=\; \mathcal{O}(\varepsilon^{-1/2}).$$

The claim follows now directly by utilizing that

$$S_{ri}(h)^{-1}\tilde{u}_i^h - S_{ri}(\bar{h})^{-1}\tilde{u}_i^{\bar{h}} \;=\; \int_0^1 D_h\left(S_{ri}(\bar{h}+s(h-\bar{h}))^{-1}\tilde{u}_i^{\bar{h}+s(h-\bar{h})}\right)(h-\bar{h})\,\mathrm{d}s. \qquad \square$$

Proof of Theorem 3.19. Let $\xi, h \in \Omega_{\rho+\delta}$ and $r \in \{1, \dots, d\}$. For simplicity of presentation, we define the maps

$$\gamma_r(h) \;\coloneqq\; S_{ri}^{-1}(h)\tilde{u}_i^h \quad \text{and} \quad \Delta(\xi, h) \;\coloneqq\; \gamma_r(\xi) - \gamma_r(h).$$

By Lemma 3.18, we obtain for $t \leq \tau$

$$\xi_r(t) - h_r(t) \;\leq\; c\varepsilon^{-2-d}\delta_\varepsilon^2\, t + \int_0^t \langle \Delta(\xi, h) + \mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon), \circ\,\mathrm{d}W\rangle.$$

For the application of the Burkholder–Davis–Gundy inequality, we need the martingale property of the stochastic integral. Hence, we transform the Stratonovich integral into an Itô integral. We write

$$\int_0^t \langle \Delta(\xi, h) + \mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon), \circ\,\mathrm{d}W\rangle \;=\; \int_0^t I(\xi) - I(h)\,\mathrm{d}s + \int_0^t \langle \Delta(\xi, h) + \mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon), \mathrm{d}W\rangle,$$

where I collects all Itô-Stratonovich correction terms. By the definition (3.20) of the drift term b, one easily verifies that these are given by

$$I(\xi) \;=\; \sum_{i,j,k=1}^{d} A_{ri}^{-1}\left[\frac{1}{2}\langle v, \psi_{i,jk}^\xi\rangle - \langle \psi_{j,k}^\xi, \tilde{u}_i^\xi\rangle - \frac{1}{2}\langle \psi_k^\xi, \tilde{u}_{ij}^\xi\rangle\right]\langle \mathcal{Q}\sigma_i^\xi, \sigma_j^\xi\rangle + \sum_{i,j=1}^{d} A_{ri}^{-1}\langle \mathcal{Q}\psi_{i,j}^\xi, \sigma_j^\xi\rangle.$$

In this expression, all terms depending on v arise from the Itô correction of the $\mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon)$-term. Moreover, since the correction $I(\xi)$ contains exactly the terms of the drift b without the critical term $\sum_i A_{ri}^{-1}\langle \mathcal{L}(v + \tilde{u}^\xi, \psi_i^\xi\rangle$, which needed a careful analysis of the nonlinear terms, we can use the estimate of Lemma 3.17 and obtain $|I(\xi)| = \mathcal{O}(\varepsilon^{-1}\delta_\varepsilon^2)$ uniformly in ξ.
So far, this shows

$$\mathbb{E}\sup_{0\leq t\leq T\wedge\tau}|\xi_r(t) - h_r(t)| \;\leq\; C\varepsilon^{-2-d}\delta_\varepsilon^2\, T + \mathbb{E}\sup_{0\leq t\leq T\wedge\tau}\left|\int_0^t \langle \Delta(\xi, h) + \mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon), \mathrm{d}W\rangle\right|. \quad (3.25)$$

By Burkholder's inequality (cf. Appendix Theorem B.14) and the Lipschitz continuity of γ from Lemma 3.20 with Lipschitz constant of order $\varepsilon^{-1/2}$, we estimate the martingale term as follows:

$$
\begin{aligned}
&\mathbb{E} \sup_{0\le t\le T\wedge\tau} \Big| \int_0^t \langle \Delta(\xi,h) + \mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon), dW\rangle \Big| \\
&\quad \le C\mathbb{E}\left[\int_0^{T\wedge\tau} \Big\langle \Delta(\xi,h) + \mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon), \mathcal{Q}\left(\Delta(\xi,h) + \mathcal{O}(\varepsilon^{-3/2-d/2}\delta_\varepsilon)\right) \Big\rangle \,\mathrm{d}s \right]^{1/2} \\
&\quad \le C\mathbb{E}\left[\int_0^{T\wedge\tau} \varepsilon^{-1}\delta_\varepsilon^2 |\xi(s)-h(s)|^2 + \varepsilon^{-2-d/2}\delta_\varepsilon^3\, |\xi(s)-h(s)| + \varepsilon^{-3-d}\delta_\varepsilon^4 \,\mathrm{d}s \right]^{1/2} \\
&\quad \le C\mathbb{E}\left[\int_0^{T\wedge\tau} \varepsilon^{-1}\delta_\varepsilon^2 |\xi(s)-h(s)|^2 + \varepsilon^{-3-d}\delta_\varepsilon^4 \,\mathrm{d}s \right]^{1/2} \\
&\quad \le C\varepsilon^{-1/2}\delta_\varepsilon T^{1/2}\, \mathbb{E} \sup_{0\le t\le T\wedge\tau} |\xi(s)-h(s)| + C\varepsilon^{-3/2-d/2}\delta_\varepsilon^2 T^{1/2} \\
&\quad \le c\varepsilon^{1/2+d/2+\ell/2}\, \mathbb{E} \sup_{0\le t\le T\wedge\tau} |\xi(s)-h(s)| + C\varepsilon^{-1/2+\ell/2}\delta_\varepsilon .
\end{aligned}
$$

In the last step, we utilized that $T < c\varepsilon^{2+d+\ell}\delta_\varepsilon^{-2}$ for some sufficiently small $c > 0$. Combined with (3.25), this yields

$$
\mathbb{E} \sup_{0\le t\le T\wedge\tau} |\xi_r(t) - h_r(t)| \le C\varepsilon^{-2-d}\delta_\varepsilon^2 T + C\varepsilon^{-1/2+\ell/2}\delta_\varepsilon \le C\varepsilon^{\ell} + C\varepsilon^{-1/2+\ell/2}\delta_\varepsilon . \qquad \square
$$

According to Theorem 3.19, the motion of the droplet's center ξ is up to times of order $\varepsilon^{2+d+\ell}\delta_\varepsilon^{-2}$ governed by the projection of the Wiener process W onto the slow manifold $\mathcal{M}_\rho$, where the error—for a sufficiently large noise strength—can be estimated by ε^ℓ. Let us briefly explain which choices of ℓ lead to a meaningful error estimate. Again by the Burkholder–Davis–Gundy inequality, we obtain for h being the projection defined by (3.24)

$$
\mathbb{E} \sup_{0\le t\le \varepsilon^{2+d+\ell}\delta_\varepsilon^{-2}} |h(t)| \sim \varepsilon^{1+d/2+\ell/2}.
$$

In order to achieve a reasonable error estimate in Theorem 3.19, we need $\varepsilon^\ell < \varepsilon^{1+d/2+\ell/2}$, or equivalently $\ell > 2+d$. Hence, the proper time scale in Theorem 3.19 is $\varepsilon^{4+2d}\delta_\varepsilon^{-2}$ and the droplet is expected to move by the order of ε^{2+d}. Clearly, this is quite far away from the time scale on which the droplet reaches the boundary of the domain Ω. On a heuristic level, this corresponds to a time scale of order $\mathcal{O}(\delta_\varepsilon^{-2})$, since the process h has to cover a distance of $\mathcal{O}(1)$. This deficiency is based on the dominating deterministic term stemming from the estimate of Lemma 3.17. While this bound decisively relies on the smallness of v in H_ε^1, are the terms of the projected dynamics in (3.24) independent of v and—in the case of a space-time white noise—essentially constant in time. Hence, any improvement of the stability region will decrease the influence of the dominating deterministic term and thereby increase the time scale where the dynamics is well approximated by the projection of the Wiener process onto the slow manifold $\mathcal{M}_\rho$.

Remark 3.21. We have seen that the droplet's motion is approximated by the projection onto the slow manifold $\mathcal{M}_\rho$. Let us discuss the matrix S^{-1} and its consequences for the dynamics in more detail. By (3.16), for a sufficiently small radius ρ the matrix S^{-1} is almost diagonal in ρ and we have

$$
S^{-1} = C_0^{-2}\rho^{-d}\left[\mathrm{I}_d + \mathcal{O}(\rho^{d-1})\right].
$$

With $\|\partial_\xi \tilde{u}^\xi\| = C_0\rho^{d/2} + \mathcal{O}(\rho^{d-1/2})$ and the approximation result of Theorem 3.19, we thus obtain that the motion of the droplet's center is in first approximation given by

$$d\xi(t) \approx \langle \mathcal{O}_{H^{-1}}(\rho^{-d/2}), \circ\, dW\rangle.$$

By Lemma 3.18, we observe that all error terms are small in ε. Therefore, as long as v stays sufficiently small, we expect that smaller droplets move faster.

3.3 Stochastic Stability

In this paragraph, we show that solutions stay close to the slow manifold $\mathcal{M}_\rho$ in H^1_ε for very long times. Here, it is convenient to work in the weighted Sobolev space H^1_ε , as it can be linked to the linearized Cahn–Hilliard operator (cf. Lemma 3.22). Moreover, it allows us to handle the nonlinear terms. In our stability analysis, we follow the method from the works of Bates and Xun (cf. [BX94, BX95]) for the one-dimensional deterministic Cahn–Hilliard equation very closely. This method was also adapted to discuss stability of fronts in the one-dimensional stochastic version in [ABK12]. For $\tilde{u}^\xi \in \mathcal{M}_\rho$, we define the functional

$$\mathcal{A}_\varepsilon(v) := \int_\Omega \varepsilon^2|\nabla v|^2 + f'(\tilde{u}^\xi)v^2\,\mathrm{d}x = \langle -\mathcal{L}^\xi v, v\rangle_{H_0^{-1}}. \tag{3.26}$$

The following lemma deals with establishing an important estimate between the functional $\mathcal{A}_\varepsilon$ and the H^1_ε-norm defined earlier in (3.21). The proof heavily relies on the spectral gaps of Theorems 3.9 and 3.12. Especially, any improvement of the spectral gap for the Cahn–Hilliard operator in three space dimensions will improve the following lemma and thereby the stability region of our main stability result in Theorem 3.23.

Lemma 3.22. *Suppose that $\xi \in \Omega_{\rho+\delta}$ and $v \perp \psi_i^\xi$ for $i = 1, \ldots, d$, where the eigenfunctions ψ_i^ξ are given by Theorem 3.9 (iii). Then, for some constants $c_1, c_2 > 0$ independent of ε, it holds true that*

$$\|v\|^2_{H^1_\varepsilon} - \mathcal{O}(\exp) \le c_1\varepsilon^{-2}\mathcal{A}_\varepsilon(v) \le c_2\varepsilon^{-(d+1)}\|\mathcal{L}^\xi v\|^2_{H^{-1}}.$$

Proof. By Theorem 3.12 we have

$$\mathcal{A}_\varepsilon(v) = \langle -\mathcal{L}^\xi v, v\rangle_{H_0^{-1}} = \langle -\mathcal{A}^\xi v, v\rangle_{L^2} > C\varepsilon^2\|v\|^2_{L^2} + \mathcal{O}(\exp),$$

where $\mathcal{A}^\xi$ denotes the linearized Allen–Cahn operator. Here, the exponentially small term arises from the fact that $\|\psi_i^\xi - w_i\|_{L^2} = \mathcal{O}(\exp)$ for $i \in \{1, \ldots, d\}$, where w_i are the eigenfunctions of $\mathcal{A}^\xi$ corresponding to the d exponentially small eigenvalues. Applying Lemma 2.10 shows that we have to introduce the $\mathcal{O}(\exp)$-term.

Let $\gamma \in (0, 1)$. Using that v is orthogonal to constants, we obtain

$$\begin{aligned}\mathcal{A}_\varepsilon(v) &> C(1-\gamma)\varepsilon^2\|v\|^2_{L^2} + \mathcal{O}(\exp) + \gamma\langle -\mathcal{A}^\xi v, v\rangle_{L^2} \\ &\ge C(1-\gamma)\varepsilon^2\|v\|^2_{L^2} + \mathcal{O}(\exp) + \gamma\varepsilon^2\|\nabla v\|^2_{L^2} - \gamma\|f'(\tilde{u}^\xi)\|_\infty\|v\|^2_{L^2}.\end{aligned}$$

In this inequality, we choose $\gamma = c\varepsilon^2$ for some sufficiently small constant $c > 0$. This choice immediately yields that $\mathcal{A}_\varepsilon(v) > c\varepsilon^2\|v\|^2_{H^1_\varepsilon} + \mathcal{O}(\exp)$.

It remains to show that

$$\mathcal{A}_\varepsilon(v) \le c\varepsilon^{1-d}\|\mathcal{L}^\xi v\|^2_{H^{-1}}.$$

We complete $\{\psi_1^\xi, \ldots, \psi_d^\xi\}$ to an orthonormal H^{-1}-basis of eigenfunctions of the operator $-\mathcal{L}^\xi$, i.e., $-\mathcal{L}^\xi\psi_i^\xi = \lambda_i\psi_i^\xi$. Note that Theorem 3.9 implies $\lambda_{d+1} > a\varepsilon^{d-1}$ for some constant $a > 0$.

As $v \perp \psi_i^\xi$ for $i = 1, \ldots, d$, we have $v = \sum_{k=d+1}^\infty \alpha_k \psi_k^\xi$ for some $\alpha_k \in \mathbb{R}$ and thus compute

$$\begin{aligned}\mathcal{A}_\varepsilon(v) &= \langle -\mathcal{L}^\xi v, v\rangle_{H_0^{-1}} = \Big\langle \sum_{k\geq d+1} \alpha_k \lambda_k \psi_k^\xi, \sum_{k\geq d+1} \alpha_k \psi_k^\xi \Big\rangle \\ &= \sum_{k\geq d+1} \alpha_k^2 \lambda_k \leq \frac{1}{\lambda_{d+1}} \sum_{k\geq d+1} \alpha_k^2 \lambda_k^2 = \frac{1}{\lambda_{d+1}} \|\mathcal{L}^\xi v\|_{H^{-1}}^2 \leq \frac{1}{a} \varepsilon^{1-d} \|\mathcal{L}^\xi v\|_{H^{-1}}^2. \qquad \square\end{aligned}$$

We can now formulate the main stability result.

Theorem 3.23 (H_ε^1-Stability)**.**
Define the stopping time

$$\tau^* := \inf\Big\{ t \in [0, T_\varepsilon] \;:\; \xi(t) \notin \Omega_{\rho+\delta} \quad \text{or} \quad \mathcal{A}_\varepsilon(v(t)) > \delta_\varepsilon^{2-\kappa} \varepsilon^{1-d} \Big\}$$

with a deterministic cut-off $T_\varepsilon = \varepsilon^{-q}$ for any large $q > 0$. We set $\tau^* = T_\varepsilon$ if none of the conditions are fulfilled for all times $t \in [0, T_\varepsilon]$. Furthermore, assume that the initial condition $v(0)$ satisfies for some constant $c > 0$

$$\mathcal{A}_\varepsilon(v(0)) < c\delta_\varepsilon^2.$$

Then, for any $\ell > 0$ there exists $C_\ell > 0$ such that

$$\mathbb{P}\Big(\mathcal{A}_\varepsilon(v(\tau^*)) > \delta_\varepsilon^{2-\kappa} \varepsilon^{1-d} \Big) < C_\ell \varepsilon^\ell.$$

Therefore, the probability that the solution exits the tube Γ' without the droplet reaching the boundary of Ω before time T_ε is smaller than any power of ε.

Proof. In the proof, we follow Section 3.6 of [ABK12] and the general framework and proof of Theorem 2.14 closely. We bound powers of $\mathcal{A}_\varepsilon(v)$ and, based on an induction argument, estimate the expectation $\mathbb{E}\mathcal{A}_\varepsilon(v(\tau^*))^p$ for arbitrary large integers p. To close the argument, we use Chebyshev's inequality.
First, we derive with Itô calculus

$$d\mathcal{A}_\varepsilon(v) = d\langle -\mathcal{L}^\xi v, v\rangle = 2\langle -\mathcal{L}^\xi v, dv\rangle + \langle -\mathcal{L}^\xi dv, dv\rangle + dR, \tag{3.27}$$

with

$$dR = \int_\Omega v^2 f''(\tilde{u}^\xi) d\tilde{u}^\xi \,\mathrm{d}x + \frac{1}{2}\int_\Omega v^2 f'''(\tilde{u}^\xi)(d\tilde{u}^\xi)^2 \,\mathrm{d}x + \int_\Omega 2v f''(\tilde{u}^\xi) dv\, d\tilde{u}^\xi \,\mathrm{d}x. \tag{3.28}$$

The terms in R appear as $\mathcal{A}_\varepsilon(v)$ depends on ξ via $f'(\tilde{u}^\xi)$. For the equation of the normal component v recall (2.13), with $\mathcal{L}(\tilde{u}^\xi) = \sum c_j^\xi u_j^\xi = \mathcal{O}(\exp)$ by Theorem 3.7,

$$dv = \Big(\sum_j c_j^\xi u_j^\xi + \mathcal{L}^\xi v + \mathcal{N}(\tilde{u}^\xi, v)\Big) dt + dW - \sum_j \tilde{u}_j^\xi d\xi_j - \frac{1}{2}\sum_{i,j} \tilde{u}_{ij}^\xi \langle \mathcal{Q}\sigma_j^\xi, \sigma_i^\xi\rangle\, dt. \tag{3.29}$$

Plugging this into (3.28) we obtain

$$\begin{aligned}dR &= \sum_j \int_\Omega v^2 f''(\tilde{u}^\xi)\tilde{u}_j^\xi \,\mathrm{d}x\, b_j(\xi)\, dt + \sum_j \int_\Omega v^2 f''(\tilde{u}^\xi)\tilde{u}_j^\xi \,\mathrm{d}x\, \langle \sigma_j, dW\rangle \\ &\quad + \sum_{i,j} \frac{1}{2} \int_\Omega v^2 f'''(\tilde{u}^\xi)\tilde{u}_j^\xi \tilde{u}_i^\xi \,\mathrm{d}x\, \langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt + \sum_{i,j} \int_\Omega v^2 f''(\tilde{u}^\xi)\tilde{u}_{ij}^\xi \,\mathrm{d}x \langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt \\ &\quad + 2\sum_{i,j} \int_\Omega v f''(\tilde{u}^\xi)\tilde{u}_i^\xi \tilde{u}_j^\xi \,\mathrm{d}x\, \langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt + 2\sum_i \int_\Omega v f''(\tilde{u}^\xi)\tilde{u}_i^\xi \mathcal{Q}\sigma_j \,\mathrm{d}x\, dt.\end{aligned}$$

To control the term dR, we use the estimates $\|\tilde{u}^\xi_j\|_\infty = \mathcal{O}(\varepsilon^{-1})$, $\|\tilde{u}^\xi_{ij}\|_\infty = \mathcal{O}(\varepsilon^{-2})$, and $\|\tilde{u}^\xi_j\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2})$, which we prove later in Lemma 3.33. Moreover, as $\tilde{u}^\xi$ is uniformly bounded, we can bound the nonlinearity f and its derivatives uniformly by a constant. By the estimates of Lemma 3.17, the drift term b is estimated by

$$|b(\xi)| = \mathcal{O}\left(\varepsilon^{-1}\delta_\varepsilon^2 + \varepsilon^{-1}\left[\|v\|^2_{H^1_\varepsilon} + \varepsilon^{-d/2}\|v\|^3_{H^1_\varepsilon}\right]\right).$$

Also, note that by definition $\|v\|_{L^2} \le \|v\|_{H^1_\varepsilon}$. We obtain

$$dR = \mathcal{O}\left((\varepsilon^{-2}\|v\|^2_{H^1_\varepsilon} + +\varepsilon^{-2}\|v\|_{H^1_\varepsilon})\delta_\varepsilon^2 + \varepsilon^{-2}\|v\|^3_{H^1_\varepsilon} + \varepsilon^{-1-d/2}\|v\|^5_{H^1_\varepsilon}\right) dt + \langle I_R, dW\rangle,$$

where I_R is defined by

$$I_R := \sum_j \int_\Omega v^2 f''(\tilde{u}^\xi)\tilde{u}^\xi_j \,\mathrm{d}x\, \sigma_j.$$

With Lemma 3.22 and $\mathcal{A}_\varepsilon(v(t)) < \delta_\varepsilon^2\varepsilon^{1-d}$ for $t \le \tau^*$, we obtain

$$\begin{aligned}
&\varepsilon^{-2}\|v\|^2_{H^1_\varepsilon}\delta_\varepsilon^2 + \varepsilon^{-2}\|v\|^3_{H^1_\varepsilon} + \varepsilon^{-1-d/2}\|v\|^5_{H^1_\varepsilon} + \varepsilon^{-2}\|v\|_{H^1_\varepsilon}\delta_\varepsilon^2 + \varepsilon^{-3/2}\|v\|_{H^1_\varepsilon}\delta_\varepsilon^2 \\
&\le C\delta_\varepsilon^2\left(\varepsilon^{-4}\mathcal{A}_\varepsilon(v) + \varepsilon^{-3}\mathcal{A}_\varepsilon(v)^{1/2}\right) + C\mathcal{A}_\varepsilon(v)\left(\varepsilon^{-5}\mathcal{A}_\varepsilon(v)^{1/2} + \varepsilon^{-6-d/2}\mathcal{A}_\varepsilon(v)^{3/2}\right) \\
&\le C\delta_\varepsilon^2\left(\varepsilon^{-3-d}\delta_\varepsilon^2 + \varepsilon^{-7/2-3d/2}\delta_\varepsilon + \varepsilon^{-7/2-3d}\delta_\varepsilon^3\right).
\end{aligned}$$

We have $\delta_\varepsilon < \varepsilon^{7/2+3d/2}$ by Assumption 3.2 and thus the term in the bracket is bounded by $\mathcal{O}(1)$. Therefore, for $t \le \tau^*$

$$dR = \mathcal{O}(\delta_\varepsilon^2)\, dt + \langle I_R, dW\rangle.$$

As a next step, we analyze the remaining terms in (3.27). With (3.29) we arrive at

$$\begin{aligned}
d\mathcal{A}_\varepsilon(v) - dR = {} & 2\langle -\mathcal{L}^\xi v, \mathcal{L}(\tilde{u}^\xi + v)\rangle\, dt - 2\sum_j \langle -\mathcal{L}^\xi v, \tilde{u}^\xi_j\rangle\, b_j(\xi)\, dt \\
& - \sum_{i,j}\langle -\mathcal{L}^\xi v, \tilde{u}^\xi_{ij}\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt + \mathrm{trace}_{H^{-1}}(-\mathcal{Q}^{1/2}\mathcal{L}^\xi\mathcal{Q}^{1/2})\, dt \\
& + \sum_{i,j}\langle -\mathcal{L}^\xi\tilde{u}^\xi_i, \tilde{u}^\xi_j\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt + \sum_i \langle -\mathcal{L}^\xi\tilde{u}^\xi_i, \mathcal{Q}\sigma_i\rangle\, dt \\
& + 2\langle -\mathcal{L}^\xi v, dW\rangle + 2\sum_j \langle \mathcal{L}^\xi v, \tilde{u}^\xi_j\rangle\langle \sigma_j, dW\rangle.
\end{aligned}$$

By Theorem 3.7, we have $\mathcal{L}(\tilde{u}^\xi) = \sum_i c_i u^\xi_i$, where the coefficients c_i and its derivatives $c_{i,j}$ with respect to any ξ_j are all exponentially small. Differentiating $\mathcal{L}(\tilde{u}^\xi)$ with respect to ξ_j yields $\mathcal{L}^\xi\tilde{u}^\xi_j = \sum_i c_{i,j}u^\xi_i + c_i u^\xi_{ij} = \mathcal{O}(\exp)$. Since $\mathcal{L}^\xi$ defines a self-adjoint operator on H^{-1}_0, we obtain

$$\langle -\mathcal{L}^\xi v, \tilde{u}^\xi_i\rangle = \langle v, -\mathcal{L}^\xi[\tilde{u}^\xi_i]\rangle = \mathcal{O}(\exp)\|v\|$$

and by the same argument, the inner products $\langle -\mathcal{L}^\xi\tilde{u}^\xi_i, \tilde{u}^\xi_j\rangle$ and $\langle -\mathcal{L}^\xi\tilde{u}^\xi_i, \mathcal{Q}\sigma_i\rangle$ are exponentially small as well. Therefore, most of the terms in (3.30) are exponentially small. Moreover, by Cauchy–Schwarz and Young's inequality together with the estimate $\|\tilde{u}^\xi_{ij}\| = \mathcal{O}(\varepsilon^{-1/2})$ from Lemma 3.33, we obtain

$$|\langle -\mathcal{L}^\xi v, \tilde{u}^\xi_{ij}\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle| \le C\delta_\varepsilon^2\varepsilon^{-1/2}\|\mathcal{L}^\xi v\|_{H^{-1}} \le \varepsilon^{-1}\delta_\varepsilon^2\|\mathcal{L}^\xi v\|^2_{H^{-1}} + C\delta_\varepsilon^2.$$

Next, we study the term

$$\langle -\mathcal{L}^\xi v, \mathcal{L}(\tilde{u}^\xi + v)\rangle = -\|\mathcal{L}^\xi v\|^2_{H^{-1}} + \mathcal{O}(\exp)\|\mathcal{L}^\xi v\|_{H^{-1}} + \langle -\mathcal{L}^\xi v, \mathcal{N}^\xi(v)\rangle.$$

In order to control the nonlinearity, we interpolate the L^p-norms between L^2 and H^1 and obtain by Nirenberg's inequality

$$\begin{aligned}\|\mathcal{N}^\xi(v)\|_{H^{-1}} &= \|3\tilde{u}^\xi v^2 + v^3\|_{L^2} \le c\|v\|_{L^4}^2 + \|v\|_{L^6}^3 \\ &\le c\|v\|_{L^2}^{2-d/2}\|\nabla v\|_{L^2}^{d/2} + c\|v\|_{L^2}^{3-d}\|\nabla v\|_{L^2}^{d} \le c\varepsilon^{-d/2}\|v\|_{H^1_\varepsilon}^2 + c\varepsilon^{-d}\|v\|_{H^1_\varepsilon}^3.\end{aligned}$$

Here, we used that by definition of the H^1_ε-norm $\|v\|_{L^2} \le \|v\|_{H^1_\varepsilon}$ and $\|\nabla v\|_{L^2} \le \varepsilon^{-1}\|v\|_{H^1_\varepsilon}$. Thus, we obtain

$$\begin{aligned}\langle -\mathcal{L}^\xi v, \mathcal{L}(\tilde{u}^\xi + v)\rangle &= -\|\mathcal{L}^\xi v\|_{H^{-1}}^2 + \mathcal{O}(\exp)\|\mathcal{L}^\xi v\|_{H^{-1}} + \langle -\mathcal{L}^\xi v, \mathcal{N}^\xi(v)\rangle \\ &\le -\frac{2}{3}\|\mathcal{L}^\xi v\|_{H^{-1}}^2 + c\left[\varepsilon^{-d/2}\|v\|_{H^1_\varepsilon}^2 + \varepsilon^{-d}\|v\|_{H^1_\varepsilon}^3\right]\|\mathcal{L}^\xi v\|_{H^{-1}} + \mathcal{O}(\exp) \\ &\le -\frac{2}{3}\|\mathcal{L}^\xi v\|_{H^{-1}}^2 + c\left[\varepsilon^{-d-1/2}\|v\|_{H^1_\varepsilon} + \varepsilon^{-3d/2-1/2}\|v\|_{H^1_\varepsilon}^2\right]\|\mathcal{L}^\xi v\|_{H^{-1}}^2 + \mathcal{O}(\exp).\end{aligned}$$

In the last step, we utilized that by Lemma 3.22

$$\|v\|_{H^1_\varepsilon}^2 \le c_1\varepsilon^{-2}\mathcal{A}_\varepsilon(v) \le c_2\varepsilon^{-(1+d)}\|\mathcal{L}^\xi v\|_{H^{-1}}^2.$$

For $t \le \tau^*$ we have

$$\begin{aligned}\varepsilon^{-d-1/2}\|v\|_{H^1_\varepsilon} + \varepsilon^{-3d/2-1/2}\|v\|_{H^1_\varepsilon}^2 &\le c\varepsilon^{-d-3/2}\mathcal{A}_\varepsilon(v)^{1/2} + c\varepsilon^{-3d/2-5/2}\mathcal{A}_\varepsilon(v) \\ &\le \varepsilon^{-d-3/2}\delta_\varepsilon + \varepsilon^{-3d/2-5/2}\delta_\varepsilon^2 \le c\varepsilon^{2+d/2} + c\varepsilon^{9/2+3d/2}\end{aligned}$$

and therefore, we showed that

$$2\langle -\mathcal{L}^\xi v, \mathcal{L}(\tilde{u}^\xi + v)\rangle \le -\frac{1}{2}\|\mathcal{L}^\xi v\|_{H^{-1}}^2 - c\varepsilon^{d-1}\mathcal{A}_\varepsilon(v) + \mathcal{O}(\exp). \tag{3.30}$$

Here, we utilized that $\|\mathcal{L}^\xi v\|_{H^{-1}}^2 \ge c\varepsilon^{d-1}\mathcal{A}_\varepsilon(v)$ by Lemma 3.22. Finally, we have to bound the trace of $-\mathcal{Q}^{1/2}\mathcal{L}^\xi\mathcal{Q}^{1/2}$. Using the uniform bound on $f'(\tilde{u}^\xi)$, we obtain

$$\begin{aligned}\operatorname{trace}(-\mathcal{Q}^{1/2}\mathcal{L}^\xi\mathcal{Q}^{1/2}) &= \sum_{k\in\mathbb{N}}\alpha_k^2\langle -\mathcal{L}^\xi e_k, e_k\rangle = \sum_{k\in\mathbb{N}}\alpha_k^2\int_\Omega \varepsilon^2|\nabla e_k|^2 + f'(\tilde{u}^\xi)e_k^2\,\mathrm{d}x \\ &\le C\sum_{k\in\mathbb{N}}\alpha_k^2\int_\Omega \varepsilon^2|\nabla e_k|^2 + e_k^2\,\mathrm{d}x = C\sum_{k\in\mathbb{N}}\alpha_k^2\|e_k\|_{H^1_\varepsilon}^2 \le C\delta_\varepsilon^2.\end{aligned}$$

In the last inequality, we made use of the proposed regularity of the Wiener process W in Assumption 3.2. Combining all the estimates yields

$$d\mathcal{A}_\varepsilon(v) = C\delta_\varepsilon^2\,dt - \left[\frac{1}{2}\|\mathcal{L}^\xi v\|_{H^{-1}}^2 + c\varepsilon^{d-1}\mathcal{A}_\varepsilon(v)\right]dt + \langle I, dW\rangle, \tag{3.31}$$

where

$$I = -2\mathcal{L}^\xi v + 2\sum_j\langle -\mathcal{L}^\xi v, \tilde{u}_j^\xi\rangle\sigma_j + I_R \quad\text{with}\quad I_R = \mathcal{O}(\varepsilon^{-1}\|v\|_{H^1_\varepsilon}^2).$$

As before, we have $\langle -\mathcal{L}^\xi v, \tilde{u}_j^\xi\rangle\sigma_j = \mathcal{O}(\exp)$ and therefore, we can bound I as follows:

$$\begin{aligned}\langle I, \mathcal{Q}I\rangle &= \mathcal{O}(\exp) + 4\langle -\mathcal{L}^\xi v, \mathcal{Q}I_R\rangle + 4\langle \mathcal{L}^\xi v, \mathcal{Q}\mathcal{L}^\xi v\rangle + \langle I_R, \mathcal{Q}I_R\rangle \\ &\le \mathcal{O}(\exp) + C\varepsilon^{-1}\delta_\varepsilon^2\|v\|_{H^1_\varepsilon}^2\|\mathcal{L}^\xi v\|_{H^{-1}} + C\delta_\varepsilon^2\|\mathcal{L}^\xi v\|_{H^{-1}}^2 + C\varepsilon^{-2}\delta_\varepsilon^2\|v\|_{H^1_\varepsilon}^4 \\ &\le \mathcal{O}(\exp) + C\delta_\varepsilon^2\|\mathcal{L}^\xi v\|^2.\end{aligned}$$

In the last step, we used that by Lemma 3.22 and the assumption $\delta_\varepsilon < \varepsilon^{7/2+3d/2}$ on the noise strength

$$\varepsilon^{-2}\|v\|_{H^1_\varepsilon}^4 \le c\varepsilon^{-6}\mathcal{A}_\varepsilon(v)^2 \le c\varepsilon^{-7-d}\mathcal{A}_\varepsilon(v)\|\mathcal{L}^\xi v\|^2 \le c\varepsilon^{-8-2d}\delta_\varepsilon^2\|\mathcal{L}^\xi v\|^2 < \varepsilon^{d-1}\|\mathcal{L}^\xi v\|^2.$$

We can now bound powers of $\mathcal{A}_\varepsilon(v)$. With Itô calculus we obtain for any $p \geq 2$

$$\begin{aligned}\frac{1}{p} d\mathcal{A}_\varepsilon(v)^p &= \mathcal{A}_\varepsilon(v)^{p-1} d\mathcal{A}_\varepsilon(v) + \frac{p-1}{2}\mathcal{A}_\varepsilon(v)^{p-2}(d\mathcal{A}_\varepsilon(v))^2 \\ &\leq C\delta_\varepsilon^2 \mathcal{A}_\varepsilon(v)^{p-1}\,dt - \frac{1}{2}\|\mathcal{L}^\xi v\|_{H^{-1}}^2 \mathcal{A}_\varepsilon(v)^{p-1}\,dt - a\varepsilon^{d-1}\mathcal{A}_\varepsilon(v)^p\,dt \\ &\quad + \mathcal{A}_\varepsilon(v)^{p-1}\langle I, dW\rangle + \frac{p-1}{2}\mathcal{A}_\varepsilon(v)^{p-2}\langle I, QI\rangle\,dt.\end{aligned}$$

Taking integrals up to the stopping time $\tau^* \leq T_\varepsilon$ and using that the expectation of a stopped stochastic integral is zero since the stopping time is deterministically bounded, we obtain for any $p \geq 2$

$$\begin{aligned}&\frac{1}{p}\mathbb{E}\mathcal{A}_\varepsilon(v(\tau^*))^p + \frac{1}{2}\mathbb{E}\int_0^{\tau^*} \mathcal{A}_\varepsilon(v)^{p-1}\|\mathcal{L}^\xi v\|_{H^{-1}}^2\,\mathrm{d}t + a\varepsilon^{d-1}\mathbb{E}\int_0^{\tau^*}\mathcal{A}_\varepsilon(v)^p\,\mathrm{d}t \\ &\leq \frac{1}{p}\mathcal{A}_\varepsilon(v(0))^p + C\delta_\varepsilon^2\,\mathbb{E}\int_0^{\tau^*}\mathcal{A}_\varepsilon(v)^{p-1}\,\mathrm{d}t \\ &\quad + \mathcal{O}(\exp)\,\mathbb{E}\int_0^{\tau^*}\mathcal{A}_\varepsilon(v)^{p-2}\,\mathrm{d}t + C\delta_\varepsilon^2\mathbb{E}\int_0^{\tau^*}\mathcal{A}_\varepsilon(v)^{p-2}\|\mathcal{L}^\xi v\|_{H^{-1}}^2\,\mathrm{d}t.\end{aligned}$$

Inductively, we see that

$$\begin{aligned}&\frac{1}{p}\mathbb{E}\mathcal{A}_\varepsilon(v(\tau^*))^p + \frac{1}{2}\mathbb{E}\int_0^{\tau^*} \mathcal{A}_\varepsilon(v)^{p-1}\|\mathcal{L}^\xi v\|_{H^{-1}}^2\,\mathrm{d}t + a\varepsilon^{d-1}\mathbb{E}\int_0^{\tau^*}\mathcal{A}_\varepsilon(v)^p\,\mathrm{d}t \\ &\leq \sum_{i=2}^p \left(\delta_\varepsilon^2\varepsilon^{1-d}\right)^{p-i}\mathcal{A}_\varepsilon(v(0))^i + C\left(\delta_\varepsilon^2\varepsilon^{1-d}\right)^{p-1}\left[\mathbb{E}\int_0^{\tau^*}\|\mathcal{L}^\xi v\|_{H^{-1}}^2\,\mathrm{d}t + \mathbb{E}\int_0^{\tau^*}\mathcal{A}_\varepsilon(v)\,\mathrm{d}t\right].\end{aligned}$$

We utilize now that by (3.31)

$$\mathbb{E}\mathcal{A}_\varepsilon(v(\tau^*)) + \frac{1}{2}\mathbb{E}\int_0^{\tau^*}\|\mathcal{L}^\xi v\|^2\,\mathrm{d}t + a\varepsilon^{d-1}\mathbb{E}\int_0^{\tau^*}\mathcal{A}_\varepsilon(v)\,\mathrm{d}t \;\leq\; \mathcal{A}_\varepsilon(v(0)) + C\delta_\varepsilon^2 T_\varepsilon$$

and obtain

$$\begin{aligned}&\frac{1}{p}\mathbb{E}\mathcal{A}_\varepsilon(v(\tau^*))^p + \frac{1}{2}\mathbb{E}\int_0^{\tau^*} \mathcal{A}_\varepsilon(v)^{p-1}\|\mathcal{L}^\xi v\|_{H^{-1}}^2\,\mathrm{d}t + a\varepsilon^{d-1}\mathbb{E}\int_0^{\tau^*}\mathcal{A}_\varepsilon(v)^p\,\mathrm{d}t \\ &\leq C\left(\delta_\varepsilon^2\varepsilon^{1-d}\right)^p T_\varepsilon + C\sum_{i=2}^p\left(\delta_\varepsilon^2\varepsilon^{1-d}\right)^{p-i}\mathcal{A}_\varepsilon(v(0))^i + \left(\delta_\varepsilon^2\varepsilon^{1-d}\right)^{p-1}\varepsilon^{1-d}\mathcal{A}_\varepsilon(v(0)) \\ &\leq C\left(\delta_\varepsilon^2\varepsilon^{1-d}\right)^p\left[1+T_\varepsilon\right].\end{aligned}$$

In the last step, we have to assume that the initial condition is sufficiently close to the slow manifold such that $\mathcal{A}_\varepsilon(v(0)) < \delta_\varepsilon^2$. So far, we have proved that for any $p \geq 2$

$$\mathbb{E}\mathcal{A}_\varepsilon(v(\tau^*))^p \;\leq\; C\left(\delta_\varepsilon^2\varepsilon^{1-d}\right)^p\left[1+T_\varepsilon\right].$$

Finally, we use this to show that the probability that v leaves the slow tube Γ' before a time $T_\varepsilon = \varepsilon^{-q}$ (i.e., $\tau^* = T_\varepsilon$) is very small. With Chebyshev's inequality we obtain

$$\begin{aligned}\mathbb{P}(\mathcal{A}_\varepsilon(v(\tau^*)) \;>\; \delta_\varepsilon^{2-\kappa}\varepsilon^{1-d}) &\leq \mathbb{E}\mathcal{A}_\varepsilon(v(\tau^*))^p\left(\delta_\varepsilon^{2-\kappa}\varepsilon^{1-d}\right)^{-p} \\ &\leq C\left(\frac{\delta_\varepsilon^2\varepsilon^{1-d}}{\delta_\varepsilon^{2-\kappa}\varepsilon^{1-d}}\right)^p\left[1+\varepsilon^{-q}\right] \\ &= C\delta_\varepsilon^{2\kappa p}\left[1+\varepsilon^{-q}\right].\end{aligned}$$

Now, choosing p large enough concludes the proof. □

Remark 3.24. By Lemma 3.22, we compute that for $t \leq \tau^*$

$$\|v(t)\|_{H^1_\varepsilon}^2 \;\leq\; c\varepsilon^{-2}\mathcal{A}_\varepsilon(v(t)) \;\leq\; c\varepsilon^{-1-d}\delta_\varepsilon^{2-\kappa}.$$

Hence, by the stability result of Theorem 3.23, we can guarantee that $\tilde{u}^\xi(t)+v(t)$ stays in Γ' (defined by (3.23)) for very long times unless the droplet hits the boundary. In that case, one needs to introduce a new slow manifold to study the motion along the boundary. For the mass conserving stochastic Allen–Cahn equation, the motion of almost semicircular droplets along the boundary was analyzed in [ABBK15]. We expect a similar behavior for the Cahn–Hilliard equation.

3.3.1 Extension to a general class of nonlinearites

In the preceding stability analysis, we treated for simplicity only the standard quartic potential. The aim of this section is to extend the result to more general nonlinearities. Recall that the critical nonlinear term is given by

$$\mathcal{N}^\xi(v) \;=\; -\Delta F^\xi(v) \quad \text{with} \quad F^\xi(v) \;:=\; F'(\tilde{u}^\xi) - F'(\tilde{u}^\xi + v) + F''(\tilde{u}^\xi)v. \tag{3.32}$$

In the sequel, we will always assume that the potential F is such that

$$|F^\xi(v)| \;\leq\; c\left(|v|^2 + |v|^p\right) \quad \text{for some } \; p>2. \tag{3.33}$$

That is, it behaves quadratically for small values of v and has at most polynomial growth. A typical example is F being a polynomial of degree $p+1$. Essential for controlling the stochastic ODE governing the motion of a single droplet is a bound on $\langle \mathcal{N}^\xi(v), \psi_i^\xi\rangle$ (see Lemma 3.17 for the quartic case). Under the assumption (3.33), we obtain

$$|\langle \mathcal{N}^\xi(v), \psi_i^\xi\rangle| \;\leq\; \|\psi_i^\xi\|_{L^2}\|F^\xi(v)\|_{L^2} \;\leq\; c\varepsilon^{-1/2}\left(\|v\|_{L^{2p}}^2 + \|v\|_{L^{2p}}^p\right), \tag{3.34}$$

where we used Hölder's inequality and $\|\psi_i^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2})$ by Lemma 3.33. Hence, we observe that the nonlinear term is manageable, if we establish control of the normal component v in L^{2p}. However, the fourth-order Cahn–Hilliard operator prohibits us from proving stochastic stability directly in L^{2p}. To apply the method used in the proof of Theorem 3.23, it is desirable to have the embedding $H^1(\Omega) \hookrightarrow L^{2p}(\Omega)$ at hand. By Sobolev embedding, however, this depends heavily on the space dimension d and the growth parameter p. We start with analyzing the cases, where this embedding holds true. These are $d=2$ and arbitrary $p>2$, or $d=3$ and $p \leq 3$. Hence, it is the higher powers in the three-dimensional setting which need a more careful analysis. For now, let us focus on the former case. Here, the statement of Theorem 3.23 still remains valid.

Theorem 3.25 (Extension to general nonlinearities I)**.**
Assume that (3.33) holds true with $p \in (2,\infty)$ for $d=2$, and $p \in (2,3]$ for $d=3$. For $\mathcal{A}_\varepsilon(v)$ defined by (3.26), consider the exit time

$$\tau^* \;=\; \inf\left\{t \in [0,T_\varepsilon] \;:\; \xi(t) \notin \Omega_{\rho+\delta} \quad \text{or} \quad \mathcal{A}_\varepsilon(v(t)) \;>\; \delta_\varepsilon^{2-\kappa}\varepsilon^{1-d}\right\}.$$

Furthermore, assume that the initial condition $v(0)$ satisfies for some constant $c>0$

$$\mathcal{A}_\varepsilon(v(0)) \;<\; c\delta_\varepsilon^2.$$

Then, for any $\ell > 0$ there exists $C_\ell > 0$ such that

$$\mathbb{P}\left(\mathcal{A}_\varepsilon(v(\tau^*)) \;>\; \delta_\varepsilon^{2-\kappa}\varepsilon^{1-d}\right) \;<\; C_\ell\varepsilon^\ell.$$

Proof. First, we observe that Lemma 3.22 is not affected by the change to a general nonlinearity. Moreover, most of the estimates in the proof of Theorem 3.23 remain valid. The difference lies only in the terms involving the drift term $b(\xi)$, since these depend on the nonlinearity $\mathcal{N}^\xi(v)$. We define the parameter

$$\theta = \theta(p,d) := \frac{p-1}{2p} d$$

and note that by the assumptions on (d,p) we always have $0 < \theta \le 1$. The constant θ is chosen in such a way that by the Gagliardo–Nirenberg interpolation inequality

$$\|v\|_{L^{2p}} \le C\|v\|_{L^2}^{1-\theta}\|\nabla v\|_{L^2}^{\theta}.$$

Together with (3.34), this furnishes the estimate

$$\begin{aligned}|\langle \mathcal{N}^\xi(v), \psi_i^\xi\rangle| &\le c\varepsilon^{-1/2}\left[\|v\|_{L^2}^{2(1-\theta)}\|\nabla v\|_{L^2}^{2\theta} + \|v\|_{L^2}^{p(1-\theta)}\|\nabla v\|_{L^2}^{p\theta}\right]\\ &\le c\varepsilon^{-1/2}\left[\varepsilon^{-2\theta}\|v\|_{H^1_\varepsilon}^2 + \varepsilon^{-p\theta}\|v\|_{H^1_\varepsilon}^p\right],\end{aligned}$$

where we utilized $\|\nabla v\|_{L^2} \le \varepsilon^{-1}\|v\|_{H^1_\varepsilon}$ by definition of the norm in H^1_ε. With the estimates from Lemma 3.17 this shows

$$|b(\xi)| = \mathcal{O}\left(\varepsilon^{-1}\delta_\varepsilon^2 + \varepsilon^{-1/2}\left[\varepsilon^{-2\theta}\|v\|_{H^1_\varepsilon}^2 + \varepsilon^{-p\theta}\|v\|_{H^1_\varepsilon}^p\right]\right).$$

The bound on b was used twice in the proof of Theorem 3.23. First, we needed to control the term

$$T_1 := \sum_j \int_\Omega v^2 f''(\tilde{u}^\xi)\tilde{u}_j^\xi \,\mathrm{d}x\, b_j(\xi).$$

For $t \le \tau^*$, we have by Lemma 3.22 the estimate $\|v\|_{H^1_\varepsilon} \le c\varepsilon^{-1}\mathcal{A}_\varepsilon^{1/2}(v) \le \varepsilon^{-1/2-d/2}\delta_\varepsilon^{1-\kappa/2}$ and hence,

$$\begin{aligned}|T_1| &\le c\varepsilon^{-1}\|v\|_{L^2}^2|b| \le c\varepsilon^{-2}\delta_\varepsilon^2\|v\|_{H^1_\varepsilon}^2 + c\varepsilon^{-2\theta-3/2}\|v\|_{H^1_\varepsilon}^4 + c\varepsilon^{-p\theta-3/2}\|v\|_{H^1_\varepsilon}^{p+2}\\ &\le c\delta_\varepsilon^2\left[\varepsilon^{-3-d}\delta_\varepsilon^{2-\kappa} + \varepsilon^{-2\theta-7/2-2d}\delta_\varepsilon^{2-2\kappa} + \varepsilon^{-p(\theta+1/2+d/2)-5/2-d}\delta_\varepsilon^{p-\kappa(p+2)/2}\right].\end{aligned}$$

Under the assumption $\delta_\varepsilon < \varepsilon^{7/2+3d/2}$, the bracket is bounded by $\mathcal{O}(1)$. For the first two summands this can be verified readily, for the third term we obtain (ignoring the small term in κ)

$$\varepsilon^{-p(\theta+1/2+d/2)-5/2-d}\delta_\varepsilon^p \le \varepsilon^{p(3+d-\theta)-5/2-d} = \varepsilon^{(p-2)(3+d-\theta)+7/2+d-2\theta}.$$

Clearly, since $p > 2$ and $d - 2\theta \ge 0$, this term is smaller than 1. In consequence, we established $|T_1| = \mathcal{O}(\delta_\varepsilon^2)$, which fits exactly in the proof of Theorem 3.23.

The second and last term involving the nonlinearity is $\langle -\mathcal{L}^\xi v, \mathcal{N}^\xi(v)\rangle$. Here, we obtain by interpolating L^4 and L^{2p} between L^2 and H^1_ε

$$\begin{aligned}|\langle -\mathcal{L}^\xi v, \mathcal{N}^\xi(v)\rangle| &\le \|\mathcal{L}^\xi v\|_{H^{-1}}\|F^\xi(v)\|_{L^2} \le c\|\mathcal{L}^\xi v\|_{H^{-1}}\left[\|v\|_{L^4}^2 + \|v\|_{L^{2p}}^p\right]\\ &\le c\|\mathcal{L}^\xi v\|_{H^{-1}}\left[\|v\|_{L^2}^{2-d/2}\|\nabla v\|_{L^2}^{d/2} + \|v\|_{L^2}^{p(1-\theta)-d/2}\|\nabla v\|_{L^2}^{p\theta+d/2}\right]\\ &\le c\|\mathcal{L}^\xi v\|_{H^{-1}}\left[\varepsilon^{-d/2}\|v\|_{H^1_\varepsilon}^2 + \varepsilon^{-p\theta-d/2}\|v\|_{H^1_\varepsilon}^p\right]\\ &\le c\|\mathcal{L}^\xi v\|_{H^{-1}}^2\left[\varepsilon^{-d-1/2}\|v\|_{H^1_\varepsilon} + \varepsilon^{-p\theta-d-1/2}\|v\|_{H^1_\varepsilon}^{p-1}\right].\end{aligned}$$

In the last step, we utilized that by Lemma 3.22

$$\|v\|_{H^1_\varepsilon}^2 - \mathcal{O}(\exp) \le c\varepsilon^{-2}\mathcal{A}_\varepsilon(v) \le \tfrac{c}{2a}\varepsilon^{-(d+1)}\|\mathcal{L}^\xi v\|_{H^{-1}}^2.$$

With help of the same estimate, we can bound the term in the bracket for $t \le \tau^*$.

Similarly to the term T_1, it is easily verified that it is bounded by a positive power of ε and thus we can establish the same estimate as in (3.30), namely

$$2\langle -\mathcal{L}^\xi v, \mathcal{L}(\tilde{u}^\xi + v)\rangle \le -\frac{1}{2}\|\mathcal{L}^\xi v\|_{H^{-1}}^2 - a\varepsilon^{d-1}\mathcal{A}_\varepsilon(v) + \mathcal{O}(\exp).$$

Hence, we are in the setting of the proof of Theorem 3.23 and can follow it verbatim. □

Finally, we treat the remaining cases that are not covered by Theorem 3.25. In the three-dimensional setting, we have for any p an embedding $H^2(\Omega) \hookrightarrow L^p(\Omega)$ and thus, it is sufficient to prove stochastic stability in H^2. However, the bounds on the linearized Cahn–Hilliard operator in H^2 are not sufficient to show stability. Therefore, we have to introduce a weighted space to assure good spectral properties. For this purpose, we endow the space $H^2(\Omega)$ with the norm

$$\|\varphi\|_\eta^2 := \varepsilon^\eta \|\Delta\varphi\|_{L^2}^2 + \|\varphi\|_{H_0^{-1}}^2.$$

The corresponding inner product will be denoted by $\langle\cdot,\cdot\rangle_\eta$. Here, $\eta > 0$ is a constant which will be fixed later. Note that in Theorem 3.9 we established the spectral properties of the linearization in H_0^{-1} and thus, it is a natural choice to consider this space. For establishing stochastic stability, we follow the method introduced in Section 2.3. First, we have to bound the stochastic differential (cf. Theorem 2.13)

$$d\|v\|_\eta^2 = 2\langle v, dv\rangle_\eta + \langle dv, dv\rangle_\eta.$$

The following lemma deals with bounding the quadratic form $\langle \mathcal{L}^\xi v, v\rangle_\eta$ for v being orthogonal to the exact eigenfunctions ψ_i^ξ corresponding to the $d = 3$ exponentially small eigenvalues of the linearized Cahn–Hilliard operator $\mathcal{L}^\xi v = -\varepsilon^2\Delta^2 v + \Delta f'(\tilde{u}^\xi)v$ (cf. Theorem 3.9). Recall that the spectral gap in the three-dimensional case is only of order ε^2, as opposed to ε in the case $d = 2$. Any improvement here will of course improve the following results.

Lemma 3.26. *Let $d = 3$, $\eta \ge 8$, and $v \perp_{H^{-1}} \operatorname{span}\{\psi_1^\xi, \psi_2^\xi, \psi_3^\xi\}$, where ψ_i^ξ denote the eigenfunctions associated to the exponentially small eigenvalues from Theorem 3.9. Then, we obtain*

$$\langle \mathcal{L}^\xi v, v\rangle_\eta \le -\frac{1}{4}\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 - c\varepsilon^2\|v\|_{H^{-1}}^2 - c\varepsilon^2\|v\|_{L^2}^2 - c\varepsilon^4\|\nabla v\|_{L^2}^2.$$

Proof. Let $\gamma_1, \gamma_2, \gamma_3 \ge 0$ with $\sum_i \gamma_i = 1$. First, we notice that

$$\langle \mathcal{L}^\xi v, v\rangle_{H^{-1}} = \langle \varepsilon^2\Delta v - f'(\tilde{u}^\xi)v, v\rangle_{L^2} \le -\varepsilon^2\|\nabla v\|_{L^2}^2 + c\|v\|_{L^2}^2,$$

where we performed integration by parts and used that $\tilde{u}^\xi$ is uniformly bounded.
By the main spectral estimates from Theorems 3.9 and 3.12, we derive

$$\begin{aligned}\langle \mathcal{L}^\xi v, v\rangle_{H^{-1}} &\le -c\gamma_1\varepsilon^2\|v\|_{H^{-1}}^2 - c\gamma_2\varepsilon^2\|v\|_{L^2}^2 - \gamma_3\varepsilon^2\|\nabla v\|_{L^2}^2 + c\gamma_3\|v\|_{L^2}^2 \\ &\le -c\varepsilon^2\|v\|_{H^{-1}}^2 - c\varepsilon^2\|v\|_{L^2}^2 - c\varepsilon^4\|\nabla v\|_{L^2}^2,\end{aligned} \tag{3.35}$$

where we fixed $\gamma_3 \approx \varepsilon^2$ and $\gamma_1, \gamma_2 = 1/2 - \mathcal{O}(\varepsilon^2)$ accordingly, and absorbed the positive L^2-term into its negative counterpart. We use these negative terms to control the H^2 inner product

$$\varepsilon^\eta\langle \Delta^2 v, -\varepsilon^2\Delta^2 v + \Delta f'(\tilde{u}^\xi)v\rangle = -\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + \varepsilon^\eta\int_\Omega \Delta^2 v\, \Delta f'(\tilde{u}^\xi)v\, dx.$$

Expanding the Laplacian of $f'(\tilde{u}^\xi)v$ yields

$$\Delta f'(\tilde{u}^\xi)v = 2f''(\tilde{u}^\xi)\nabla\tilde{u}^\xi\nabla v + f'(\tilde{u}^\xi)\Delta v + \left[f'''(\tilde{u}^\xi)|\nabla\tilde{u}^\xi|^2 + f''(\tilde{u}^\xi)\Delta\tilde{u}^\xi\right] v.$$

For the term linear in ∇v we obtain

$$\varepsilon^\eta \int_\Omega \Delta^2 v\, f''(\tilde u^\xi)\nabla u \nabla v \,dx \;\le\; c\varepsilon^{\eta-1}\|\Delta^2 v\|_{L^2}\|\nabla v\|_{L^2} \;\le\; \frac14 \varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-4}\|\nabla v\|_{L^2}^2,$$

where we utilized Young‘s inequality and $\|\nabla \tilde u^\xi\|_\infty = \mathcal{O}(\varepsilon^{-1})$ by Lemma 3.33. For the term involving Δv we similarly find that

$$\varepsilon^\eta \int_\Omega \Delta^2 v\, f'(\tilde u^\xi)\Delta v \,dx \;\le\; c\varepsilon^{\eta}\|\Delta^2 v\|_{L^2}^{3/2}\|v\|_{L^2}^{1/2} \;\le\; \frac14 \varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-6}\|v\|_{L^2}^2.$$

The last term is bounded by

$$\begin{aligned}\varepsilon^\eta \int_\Omega \Delta^2 v \left[f'''(\tilde u^\xi)|\nabla \tilde u^\xi|^2 + f''(\tilde u^\xi)\Delta \tilde u^\xi\right] v \,dx \;&\le\; c\varepsilon^{\eta-2}\|\Delta^2 v\|_{L^2}\|v\|_{L^2}\\ &\le\; \frac14 \varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-6}\|v\|_{L^2}^2.\end{aligned}$$

In order to absorb the preceding estimates into the good terms from inequality (3.35), we need to assume that $\eta \ge 8$. □

As the next step, we bound the inner product with the nonlinearity $\mathcal{N}^\xi(v) = -\Delta F^\xi(v)$ defined by (3.32). For a sufficiently small radius in v with respect to the H^2-norm, we can absorb the nonlinearity completely in the negative terms from the estimate of Lemma 3.26.

Lemma 3.27. *Let $d = 3$ and assume that $\|\Delta v\|_{L^2} \le c\varepsilon^2$. Then, it holds true that*

$$\langle \mathcal{N}^\xi(v), v\rangle_\eta \;\le\; c\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-2}\|v\|_{L^2}^2 + c\varepsilon^{\eta}\|\nabla v\|_{L^2}^2.$$

Proof. By assumption (3.33) and the Gagliardo–Nirenberg interpolation inequality, we obtain

$$\langle \mathcal{N}^\xi(v), v\rangle_{H^{-1}} \;=\; \langle F^\xi(v), v\rangle_{L^2} \;\le\; c\left[\|v\|_{L^3}^3 + \|v\|_{L^{p+1}}^{p+1}\right] \le c\left[\|\Delta v\|_{L^2} + \|\Delta v\|_{L^2}^{p-1}\right]\|v\|_{L^2}^2.$$

In the study of the inner product $\langle \Delta^2 v, \mathcal{N}^\xi(v)\rangle_{L^2}$, we will only treat the critical quadratic term in more detail. The analysis of higher powers can be carried out analogously and does not influence the result. For the sake of simplicity, we will also denote the prefactor of v^2, which is a smooth and bounded function in the variable $\tilde u^\xi$, by g. With that, let us estimate the inner product $\langle \Delta^2 v, -\Delta g(\tilde u^\xi)v^2\rangle_\eta$. Expanding the Laplacian of $g(\tilde u^\xi)v^2$ yields

$$\begin{aligned}\Delta g(\tilde u^\xi)v^2 \;&=\; \left[g''(\tilde u^\xi)|\nabla \tilde u^\xi|^2 + g'(\tilde u^\xi)\Delta \tilde u^\xi\right]v^2 + \left[4g'(\tilde u^\xi)\nabla \tilde u^\xi + 2g(\tilde u^\xi)\right]v\nabla v + 2g(\tilde u^\xi)|\nabla v|^2\\ &=: T_1 + T_2 + T_3.\end{aligned}$$

The first term can be estimated by

$$\begin{aligned}\varepsilon^\eta\left|\langle T_1, \Delta^2 v\rangle\right| \;&\le\; c\varepsilon^{\eta-2}\|\Delta^2 v\|_{L^2}\|v\|_{L^4}^2 \;\le\; c\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-6}\|v\|_{L^4}^4\\ &\le\; c\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-6}\|\Delta v\|_{L^2}^2\|v\|_{L^2}^2.\end{aligned}$$

Here, we used that by the Gagliardo–Nirenberg interpolation inequality $\|v\|_{L^4} \le \|v\|_{L^2}^{5/8}\|\Delta v\|_{L^2}^{3/8}$. For the second and third term, we obtain in a similar fashion

$$\varepsilon^\eta\left|\langle T_2, \Delta^2 v\rangle\right| \;\le\; c\varepsilon^{\eta-1}\|v\|_{L^4}\|\nabla v\|_{L^4}\|\Delta^2 v\|_{L^2} \;\le\; c\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-4}\|\Delta v\|_{L^2}^2\|\|\nabla v\|_{L^2}^2,$$

and

$$\varepsilon^\eta\left|\langle T_3, \Delta^2 v\rangle\right| \;\le\; c\varepsilon^{\eta}\|\nabla v\|_{L^4}^2\|\Delta^2 v\|_{L^2} \;\le\; c\varepsilon^{\eta}\|\Delta v\|_{L^2}\|\Delta^2 v\|_{L^2}^2.$$ □

By combining the estimates from Lemmata 3.26 and 3.27, we achieved the main estimate for proving stochastic stability (compare to Metatheorem 2).

Corollary 3.28. *Let $d = 3$, $\eta \geq 8$, and $v \perp_{H^{-1}} \operatorname{span}\{\psi_1^\xi, \psi_2^\xi, \psi_3^\xi\}$ with $\|v\|_\eta \leq c\varepsilon^{2+\eta/2}$. Under these assumptions, we obtain*

$$\langle \mathcal{L}(\tilde{u}^\xi + v), v\rangle_\eta \leq -c\varepsilon^2 \|v\|_\eta^2.$$

Proof. This is a direct consequence of the estimates in Lemmata 3.26 and 3.27, noting that $\|v\|_\eta \leq c\varepsilon^\eta \|\Delta^2 v\|_{L^2}$ by Poincaré's inequality. □

Following the guideline from Section 2.3, it remains to estimate the remainder of $\langle v, dv\rangle_\eta$ (Metatheorem 4) and the Itô correction $\langle dv, dv\rangle_\eta$ (Metatheorem 3). Since we are working in a weighted H^2-space, we have to propose additional spatial regularity of the Wiener process W. We assume in this case additionally that (cf. Assumption 3.2)

$$\rho_\varepsilon^2 := \varepsilon^\eta \operatorname{trace}_{L^2}(\mathcal{Q}^{1/2}\Delta^2\mathcal{Q}^{1/2}) + \operatorname{trace}_{H^{-1}}(\mathcal{Q}) = \sum_{k\in\mathbb{N}} \alpha_k^2 \|e_k\|_\eta^2 < \infty. \tag{3.36}$$

Note that our final stability result will thus depend on ρ_ε.

Lemma 3.29. *Under the assumptions of Corollary 3.28, we have*

$$\begin{aligned}\langle v, d\tilde{u}^\xi\rangle_\eta = &\left[c\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2} + c\varepsilon^2\|v\|_{L^2}^2 + c\varepsilon^{\eta-5}\rho_\varepsilon^4 + c\varepsilon^{3/2+\eta/2}\rho_\varepsilon^2\right] dt \\ &+ \langle \mathcal{O}_{H^{-1}}(\varepsilon^{\eta-5/2}\|\Delta v\|_{L^2}), dW\rangle_{H^{-1}}.\end{aligned}$$

Proof. By Itô's formula and the exact stochastic ODE (3.18) governing the motion of the droplet's center of Section 3.2.2, we derive

$$\langle v, d\tilde{u}^\xi\rangle_\eta = \sum_i \langle v, \tilde{u}_i^\xi\rangle_\eta \left[b_i(\xi, v)\, dt + \langle \sigma_i(\xi, v), dW\rangle_{H^{-1}}\right] + \sum_{i,j}\langle v, \tilde{u}_{ij}^\xi\rangle_\eta \langle \mathcal{Q}\sigma_i, \sigma_j\rangle_{H^{-1}}\, dt. \tag{3.37}$$

We start with estimating the inner products in $H^2(\Omega)$. For the second summand in (3.37), which does not involve the differential $d\xi$, we obtain

$$\varepsilon^\eta |\langle \Delta^2 v, \tilde{u}_{ij}^\xi\rangle_{L^2}||\langle \mathcal{Q}\sigma_i, \sigma_j\rangle_{H^{-1}}| \leq c\varepsilon^{\eta-3/2}\rho_\varepsilon^2\|\Delta^2 v\|_{L^2} \leq c\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-5}\rho_\varepsilon^4,$$

where we used that $\|\tilde{u}_{ij}^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-3/2})$ by Lemma 3.33 and $\|\mathcal{Q}\|_{L(H^{-1})} \leq \rho_\varepsilon^2$.
With Lemma 3.17 and the assumption (3.33) on $F^\xi(v)$, we see that the drift term $b(\xi, v)$ can be bounded by

$$|b(\xi, v)| \leq c\varepsilon^{-1}\rho_\varepsilon^2 + |\langle F^\xi(v), \psi_i^\xi\rangle_{L^2}| \leq c\varepsilon^{-1}\rho_\varepsilon^2 + c\varepsilon^{-1}\left(\|v\|_{L^2}^2 + \|v\|_{L^p}^p\right).$$

Hence, we derive

$$\begin{aligned}\varepsilon^\eta |\langle \Delta^2 v, \tilde{u}_i^\xi\rangle||b| &\leq c\varepsilon^{\eta-3/2}\|\Delta^2 v\|_{L^2}\left[\rho_\varepsilon^2 + \|v\|_{L^2}^2 + \|v\|_{L^p}^p\right] \\ &\leq c\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-5}\left[\rho_\varepsilon^4 + \|v\|_{L^2}^4 + \|v\|_{L^p}^{2p}\right] \\ &\leq c\varepsilon^{2+\eta}\|\Delta^2 v\|_{L^2}^2 + c\varepsilon^{\eta-5}\rho_\varepsilon^4 + c\varepsilon^{\eta-5}\left[\|\Delta v\|_{L^2}^2 + \|\Delta v\|_{L^2}^{2p-2}\right]\|v\|_{L^2}^2.\end{aligned}$$

In a similar fashion, the martingale term is estimated by

$$\varepsilon^\eta \langle \Delta v, \Delta\tilde{u}^\xi\rangle_{L^2}\langle \sigma_i, dW\rangle_{H^{-1}} = \langle \mathcal{O}_{H^{-1}}(\varepsilon^{\eta-5/2}\|\Delta v\|_{L^2}), dW\rangle_{H^{-1}}.$$

Finally, we turn to the inner products in $H^{-1}(\Omega)$. By Theorems 3.9 and 3.11, the relative distance of vectors in $\operatorname{span}\{\psi_1^\xi, \dots, \psi_d^\xi\}$ and $\operatorname{span}\{\tilde u_1^\xi, \dots, \tilde u_d^\xi\}$ is exponentially small. Therefore, for some $\alpha \in \mathbb{R}^3$,

$$\langle \tilde u_i^\xi, v\rangle_{H^{-1}} = \sum_j \alpha_j \langle \psi_j^\xi, v\rangle_{H^{-1}} + \mathcal{O}(\exp)\|\tilde u_j^\xi\|_{H^{-1}}\|v\|_{H^{-1}} = \mathcal{O}(\exp).$$

Since this is the prefactor of the first summand in (3.37) and the drift b and diffusion σ are uniformly bounded by a polynomial in ε^{-1}, we immediately obtain

$$\langle v, \tilde u_i^\xi\rangle_{H^{-1}} \Big[b_i(\xi, v)\, dt + \langle \sigma_i(\xi, v), dW\rangle_{H^{-1}} \Big] = \mathcal{O}(\exp)\, dt + \langle \mathcal{O}(\exp), dW\rangle_{H^{-1}}.$$

The second summand in (3.37) is estimated by

$$\langle v, \tilde u_{ij}^\xi\rangle_{H^{-1}} \langle \mathcal{Q}\sigma_i, \sigma_j\rangle_{H^{-1}}\, dt = \mathcal{O}(\varepsilon^{-1/2}\rho_\varepsilon^2)\|v\|_{H^{-1}}\, dt = \mathcal{O}(\varepsilon^{3/2+\eta/2}\rho_\varepsilon^2)\, dt,$$

where we utilized that by assumption $\|v\|_{H^{-1}} \le \|v\|_\eta \le c\varepsilon^{2+\eta/2}$ together with $\|u_{ij}^\xi\| = \mathcal{O}(\varepsilon^{-1/2})$, $\|\sigma\| = \mathcal{O}(1)$, and $\|\mathcal{Q}\|_{L(H^{-1})} \le \rho_\varepsilon^2$. □

Lemma 3.30. *Let ρ_ε^2 be the noise strength defined by (3.36). We have*

$$\langle dv, dv\rangle_\eta = \mathcal{O}(\varepsilon^{\eta-5/2}\rho_\varepsilon^2)\, dt.$$

Proof. First, we obtain

$$\begin{aligned}\langle dv, dv\rangle_{H^{-1}} &= \langle dW, dW\rangle_{H^{-1}} + \sum_{i,j} \langle \tilde u_i^\xi, \tilde u_j^\xi\rangle_{H^{-1}} \langle \sigma_i, \mathcal{Q}\sigma_j\rangle_{H^{-1}}\, dt - \sum_{i,j} \langle \tilde u_i^\xi, \mathcal{Q}\sigma_j\rangle_{H^{-1}}\, dt \\ &= \Big[\operatorname{trace}_{H^{-1}}(\mathcal{Q}) + \mathcal{O}(\|\mathcal{Q}\|_{L(H^{-1})}) \Big]\, dt = \mathcal{O}(\rho_\varepsilon^2)\, dt.\end{aligned}$$

Secondly, the H^2-term gives

$$\begin{aligned}\varepsilon^\eta \langle d\Delta v, d\Delta v\rangle_{L^2} &= \varepsilon^\eta \langle d\Delta W, d\Delta W\rangle_{L^2} + \varepsilon^\eta \sum_{i,j} \langle \Delta \tilde u_i^\xi, \Delta \tilde u_j^\xi\rangle_{L^2} \langle \sigma_i, \mathcal{Q}\sigma_j\rangle_{H^{-1}}\, dt \\ &\quad - \varepsilon^\eta \sum_{i,j} \langle \Delta \tilde u_i^\xi, d\Delta W\rangle_{L^2} \langle \sigma_j, dW\rangle_{H^{-1}} \\ &= \mathcal{O}(\varepsilon^{\eta-5}\delta_\varepsilon^2 + \delta_\varepsilon^2 + \varepsilon^{2\eta-5}\rho_\varepsilon^2)\, dt,\end{aligned}$$

where we utilized that by series expansion of $W = \sum \alpha_k \beta_k(t) e_k$

$$\begin{aligned}\varepsilon^\eta \langle \Delta \tilde u_i^\xi, d\Delta W\rangle_{L^2} \langle \sigma_j, dW\rangle_{H^{-1}} &= \varepsilon^\eta \sum_{k\in\mathbb{N}} \alpha_k \langle \Delta \tilde u_i^\xi, \Delta e_k\rangle_{L^2}\, \alpha_k \langle \sigma_j, e_k\rangle_{H^{-1}}\, dt \\ &\le \varepsilon^\eta \rho_\varepsilon \|\Delta \tilde u_i^\xi\|_{L^2}\, \rho_\varepsilon \|\sigma_j\|_{H^{-1}}\, dt \le c\varepsilon^{\eta-5/2}\rho_\varepsilon^2\, dt.\end{aligned}$$ □

Finally, we estimated every single term for the stochastic differential $d\|v\|_\eta$ and thereby furnished the analogue of the main inequality of Theorem 2.13, namely

Theorem 3.31. *Let $d = 3$, $\eta \ge 8$, and $v \perp_{H^{-1}} \operatorname{span}\{\psi_1^\xi, \psi_2^\xi, \psi_3^\xi\}$.*
As long as $\|v\|_\eta < \varepsilon^{2+\eta/2}$, we have

$$d\|v\|_\eta^2 = \Big[-c\varepsilon^2 \|v\|_\eta^2 + c\varepsilon^{\eta-5}\rho_\varepsilon^4 + c\rho_\varepsilon^2 \Big]\, dt + \langle \mathcal{O}(\|v\|_\eta), dW\rangle_\eta.$$

With that, we are in the setting of the main stability result of Theorem 2.14 and can prove that the weighted H^2-norm of the distance to the slow manifold stays indeed small for very long times with high probability.

Theorem 3.32 (Extension to general nonlinearities II)**.**
For $\eta \geq 8$ define the stopping time

$$\tau^* := \inf\Big\{t \in [0, T_\varepsilon] \,:\, \xi(t) \notin \Omega_{\rho+\delta} \quad or \quad \|v(t)\|_\eta > c\varepsilon^{2+\eta/2}\Big\}$$

with $T_\varepsilon = \varepsilon^{-N}$ for any fixed large $N > 0$. For some $0 < \nu < 1$, suppose that

$$\|v(0)\|_\eta < \nu\varepsilon^{2+\eta/2}.$$

Moreover, suppose that for some very small $\kappa > 0$ the noise strength satisfies

$$\rho_\varepsilon < \varepsilon^{3+\eta/2+\kappa}.$$

Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon)$ is smaller than any power of ε, as ε tends to zero.

Proof. We can apply the general result of Theorem 2.14. In fact, with $K_\varepsilon = \mathcal{O}(\varepsilon^{\eta-5}\rho_\varepsilon^4 + \rho_\varepsilon^2)$, $a_\varepsilon = c\varepsilon^2$, and $R_\varepsilon^2 = c\varepsilon^{4+\eta}$, we obtain

$$\frac{K_\varepsilon}{a_\varepsilon R_\varepsilon^2} \leq c\varepsilon^{-11}\rho_\varepsilon^4 + c\varepsilon^{-6-\eta}\rho_\varepsilon^2 \leq c\varepsilon^{2\kappa}. \qquad \square$$

3.4 Estimates

In this final section, we give all the estimates that were needed throughout the analysis of the stochastic Cahn–Hilliard equation. Compared to the deterministic counterpart, we need to bound higher order derivatives, which arise due to Itô calculus. We start with the estimates with respect to the L^2- and L^∞-norm.

Lemma 3.33. *For $i = 1, \ldots, d$, let ψ_i^ξ be the orthonormal basis from Theorem 3.9 and $\tilde{u}^\xi$ the bubble as constructed in Theorem 3.7. Further subindices will denote partial derivatives with respect to ξ. The following estimates hold true*

$$\|\tilde{u}_j^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2}), \qquad \|\psi_i^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2}),$$
$$\|\tilde{u}_{ij}^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-3/2}), \quad and \quad \|\psi_{i,j}^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-3/2}).$$

Moreover, in L^∞ we find that

$$\|\tilde{u}_j^\xi\|_\infty = \mathcal{O}(\varepsilon^{-1}), \quad \|\tilde{u}_{ij}^\xi\|_\infty = \mathcal{O}(\varepsilon^{-2}), \quad and \quad \|\psi_j^\xi\|_\infty = \mathcal{O}(\varepsilon^{-1}).$$

Proof. First, we observe that by Theorem 3.7 it suffices to analyze the partial derivatives of u^ξ, since the correction term v^ξ and all its derivatives are exponentially small. By Lemmata 3.5 and 3.6 we have

$$\frac{\partial u^\xi}{\partial \xi_i} = \varepsilon^{-1}\frac{\partial U^*}{\partial r}\frac{\partial r}{\partial \xi_i} + \varepsilon^{-2}\frac{\partial U^*}{\partial \rho}\frac{\partial a^\xi}{\partial \xi_i} = \left[\varepsilon^{-1}U'\left(\frac{r-\rho}{\varepsilon}\right) + \mathcal{O}(1)\right]\frac{\partial r}{\partial \xi_i} + \mathcal{O}(\exp), \tag{3.38}$$

where we defined $r = |x - \xi|$ and utilized that the partial derivatives of a^ξ with respect to ξ_i for $i = 1, \ldots, d$ are all exponentially small (cf. Lemma 3.6). We use the radial geometry of the problem and the fact that U' localizes around the boundary of the bubble. For some small $\delta > 0$, we consider the ring $\Omega_\delta := \Big\{x : \Big||x-\xi| - \rho\Big| \leq \delta\Big\}$. We compute

$$\begin{aligned}\varepsilon^{-2}\int_{\Omega_\delta} U'\left(\frac{r-\rho}{\varepsilon}\right)^2\left(\frac{\partial r}{\partial \xi_i}\right)^2 \mathrm{d}x &\leq \varepsilon^{-2}\int_{\Omega_\delta} U'\left(\frac{r-\rho}{\varepsilon}\right)^2 \mathrm{d}x \\ &\leq C\varepsilon^{-1}\int_{|\eta|\leq\delta/\varepsilon} U'(\eta)^2(\varepsilon\eta+\rho)^{d-1}\,\mathrm{d}\eta \leq C\rho^{d-1}\varepsilon^{-1}\int_{\mathbb{R}} U'(\eta)^2\,\mathrm{d}\eta \leq C\varepsilon^{-1}.\end{aligned}$$

On the set $\Omega \setminus \Omega_\delta$ we utilize $|U'(\eta)| \le ce^{-c|\eta|}$ and derive

$$\varepsilon^{-2} \int_{\Omega\setminus\Omega_\delta} U'\left(\frac{r-\rho}{\varepsilon}\right)^2 \left(\frac{\partial r}{\partial \xi_i}\right)^2 \,\mathrm{d}x \;\le\; C\varepsilon^{-2} e^{-c\delta/\varepsilon}\, |\Omega \setminus \Omega_\delta| \;=\; \mathcal{O}(\exp).$$

Combined with (3.38) this shows $\|\tilde{u}_j^\xi\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2})$ for any $j \in \{1, \dots, d\}$.

With help of the relation (3.38), we can also estimate the uniform norm of $\tilde{u}_j^\xi$. Note that the heteroclinic U is given as solution to $U'' - F'(U) = 0$, which is equivalent to the equation $U' = \sqrt{2F(U)}$. Hence, we obtain

$$\|u_i^\xi\|_\infty \;=\; \varepsilon^{-1}\|U'\|_\infty + \mathcal{O}(1) \;=\; \varepsilon^{-1} \sup_{x\in(-1,1)} \sqrt{2F(x)} + \mathcal{O}(1).$$

Estimating the second order derivatives of $\tilde{u}^\xi$ can be carried out analogously. By definition (3.8) of the eigenfunction ψ_i^ξ, we see that

$$\|\psi_i^\xi\|_{L^2} \;\le\; C\frac{\|\nabla\tilde{u}^\xi\|_{L^2}}{\|\nabla\tilde{u}^\xi\|} \;\le\; C\rho^{-d/2}\varepsilon^{-1/2},$$

where we used the previous bound on the L^2-norm and that $\|\nabla\tilde{u}^\xi\| \;=\; c\rho^{d/2} + \mathcal{O}(\rho^{d-1/2})$ by [AFK04, Theorem 3.6]. Moreover, we compute

$$\left\|\partial_j \frac{\tilde{u}_k^\xi}{\|\tilde{u}_k^\xi\|}\right\|_{L^2} \;=\; \left\|-\frac{\tilde{u}_k^\xi\langle\tilde{u}_k^\xi, \tilde{u}_{kj}^\xi\rangle}{\|\tilde{u}_k^\xi\|^3} + \frac{\tilde{u}_{kj}^\xi}{\|\tilde{u}_k^\xi\|}\right\|_{L^2} \;\le\; \frac{\|u_k^\xi\|_{L^2}\|u_{kj}^\xi\|}{\|u_k^\xi\|^2} + \frac{\|u_{kj}^\xi\|_{L^2}}{\|u_k^\xi\|} \;\le\; \frac{\|u_k^\xi\|_{L^2} + \|u_{kj}^\xi\|_{L^2}}{\|u_k^\xi\|}.$$

With the preceding estimates, it is readily verified that this term is of order $\mathcal{O}(\varepsilon^{-3/2})$. Finally, by the definition in Theorem 3.9 we derive

$$\|\psi_{i,j}^\xi\|_{L^2} \le \sum_k |\partial_j a_{ki}^\xi| \frac{\|\tilde{u}_k^\xi\|_{L^2}}{\|\tilde{u}_k^\xi\|} + \mathcal{O}(\varepsilon^{-3/2}) \;\le\; C\varepsilon^{-1/2}\sum_k |\partial_j a_{ki}^\xi| + \mathcal{O}(\varepsilon^{-3/2}) = \mathcal{O}(\varepsilon^{-3/2}), \quad (3.39)$$

where we used that the matrix (a_{ki}^ξ) does depend smoothly on ξ and is non-singular. Since $\|\tilde{u}_j^\xi\|_\infty = \mathcal{O}(\varepsilon^{-1})$, the uniform bound of $\psi_{i,j}^\xi$ is easily verified. $\square$

We conclude our analysis of the stochastic Cahn–Hilliard equation with giving all the necessary bounds on derivatives of the droplet state $\tilde{u}^\xi$ and the eigenfunctions ψ_i^ξ in H^{-1} that were used throughout this chapter. Heuristically, the H^{-1}-norm eliminates one derivative and it remains to control the „antiderivate“ in L^2. Hence, we can rely on the estimates of Lemma 3.33 in the following result.

Lemma 3.34. *Under the same assumptions as in Lemma 3.33, we have*

$$\begin{aligned} \|\tilde{u}_j^\xi\| \;&=\; \mathcal{O}(1), & \|\psi_{i,j}^\xi\| \;&=\; \mathcal{O}(\varepsilon^{-1}),\\ \|\tilde{u}_{ij}^\xi\| \;&=\; \mathcal{O}(\varepsilon^{-1/2}), \quad \textit{and} & \|\psi_{i,jk}^\xi\| \;&=\; \mathcal{O}(\varepsilon^{-3/2}). \end{aligned}$$

Proof. First, we note that the bound $\|\psi_{i,j}^\xi\| = \mathcal{O}(\varepsilon^{-1})$ was established in Theorem 6.1 in [AF98], and secondly, $\|\tilde{u}_j^\xi\| = c\rho^{d/2} + \mathcal{O}(\rho^{d+1/2})$ holds true by Theorem 3.6 in [AFK04].
By a characterization of the dual space H^{-1} (cf. [Eva10, Section 5.9, Theorem 1]), we find for $g \in H^{-1}$ functions $f_1, \dots, f_d \in L^2$ such that

$$g \;=\; \nabla\cdot f \;=\; \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_d}{\partial x_d} \quad \text{and} \quad \|g\|^2 \;=\; \inf_{g=\nabla\cdot f} \int_\Omega |f|^2 \,\mathrm{d}x.$$

Using the relation $\tilde{u}_i^\xi = \partial_{x_i} u^\xi + \mathcal{O}(\exp)$ and choosing $f_j = \partial_{x_i} u^\xi$, we have

$$\|\tilde{u}_{ij}^\xi\|^2 \;=\; \|\partial_j f_j\|^2 + \mathcal{O}(\exp) \;\leq\; \|f_j\|_{L^2}^2 + \mathcal{O}(\exp) \;=\; \mathcal{O}(\varepsilon^{-1}),$$

where we utilized that $\|f_j\|_{L^2} = \mathcal{O}(\varepsilon^{-1/2})$ by Lemma 3.33. The same argument yields

$$\|\psi_{i,jk}^\xi\| \leq \|\psi_{i,j}^\xi\|_{L^2} \;\leq\; c\varepsilon^{-3/2}.$$

□

CHAPTER 4

Droplet motion for the mass conserving stochastic Allen–Cahn equation

Mathematically closely related to the stochastic Cahn–Hilliard equation is the mass conserving version of the stochastic Allen–Cahn equation

$$\begin{cases} \partial_t u(x,t) = \varepsilon^2 \Delta u(x,t) - f(u(x,t)) + \frac{1}{|\Omega|}\int_\Omega f(u(x,t))\,\mathrm{d}x + \dot{W}(x,t), & x \in \Omega, \\ \partial_\eta u(x,t) = 0, & x \in \partial\Omega, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{SAC}$$

As in the previous chapter, we are again concerned with the higher dimensional cases and hence, $\Omega \subset \mathbb{R}^d$ is a sufficiently smooth bounded domain of area $|\Omega|$, where we allow for the space dimensions $d = 2$ and $d = 3$. Moreover, f denotes the derivative of the smooth double well potential F that we introduced in Chapter 3 for the Cahn–Hilliard equation. For simplicity, we will only focus on the standard quartic potential $F(u) = \frac{1}{4}(u^2-1)^2$, although most of the results hold for a very general class of nonlinearities. Similarly to the extension to general nonlinearites in Section 3.3.1 for the Cahn–Hilliard equation, only the precise formulation of the stability result and the condition on the noise strength do change depending on the growth of F at ∞. The stochastic forcing is given by an additive white in time noise $\partial_t W$. Similarly to Definition 3.1, W is given by a $\mathcal{Q}$-Wiener process in the Hilbert space $L^2(\Omega)$, i.e.,

$$\mathcal{Q}e_k = \alpha_k^2 e_k \qquad \text{and} \qquad W(t) = \sum_{k\in\mathbb{N}} \alpha_k \beta_k(t) e_k,$$

where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis with corresponding eigenvalues α_k^2 and $\{\beta_k\}_{k\in\mathbb{N}}$ is a family of independent real-valued standard Brownian motions. To guarantee that solutions to (SAC) preserve the total mass, we have to assume that the Wiener process takes its values in L_0^2. Moreover, as our method is based on the application of Itô formula, we assume that W is sufficiently smooth, that is, $\mathcal{Q}$ is trace-class. Throughout our analysis, we assume that the Wiener process W enjoys the following regularity properties:

Assumption 4.1 (Regularity of the Wiener process)**.**
The process W satisfies

$$\int_\Omega W(t,x)\,\mathrm{d}x = 0 \quad \textit{for any } t \geq 0.$$

Furthermore, we assume that

$$\mathrm{trace}(\mathcal{Q}) = \sum_{k\in\mathbb{N}} \alpha_k^2 =: \eta_0 < \infty \quad \textit{and} \quad \mathrm{trace}(-\mathcal{Q}^{1/2}\Delta\mathcal{Q}^{1/2}) = \sum_{k\in\mathbb{N}} \alpha_k^2 \|\nabla e_k\|_{L^2}^2 =: \eta_2 < \infty.$$

Recall that for the induced L^2 operator norm of $\mathcal{Q}$ we always have that $\|\mathcal{Q}\| \leq \mathrm{trace}(\mathcal{Q}) = \eta_0$. Also, note that e_k is normalized in L^2, but not in H^1.

The deterministic Allen–Cahn equation forms a gradient flow with respect to the same energy functional J_ε defined by (3.1) as the Cahn–Hilliard equation, but with respect to the L_0^2-topology as opposed to H^{-1}. Therefore, we can expect a very similar behavior. In fact, in our analysis we will rely on the same slow manifold of droplets.

The aim of this chapter is to analyze the motion of a single droplet for the mass conserving stochastic Allen–Cahn equation in the interior of the set Ω. We can build on the analysis for the Cahn–Hilliard equation. Moreover, in the stability analysis we will follow [ABBK15], where the slow motion of a semicircular droplet along the boundary $\partial\Omega$ was studied. In this article, the two-dimensional case was analyzed, while in our analysis we will include a three-dimensional domain.

4.1 The slow manifold $\mathcal{M}_\rho$

In this section, we briefly collect some results from Chapter 3 that will be used throughout the remainder of our study of the Allen–Cahn equation. In the construction of a slow manifold, we follow the analysis of Section 3.1 for the deterministic Cahn–Hilliard equation, which is based on the work of Alikakos and Fusco [AF98]. In Proposition 3.4, we established for a fixed radius $\rho > \bar{\rho} > 0$ the existence of a radial solution $U^*(|x|, \rho)$ to the problem

$$\Delta u - f(u) \;=\; \sigma(\rho), \quad x \in \mathbb{R}^d.$$

For a fixed small minimal distance $\delta > 0$ of the droplets to the boundary $\partial\Omega$, let ξ be in the set $\Omega_{\rho+\delta} = \{\xi : d(\xi, \partial\Omega) > \rho + \delta\}$. Analogously to Definition (3.5), we define the rescaled and translated function $u^\xi : \Omega \to \mathbb{R}$ by

$$u^\xi(x) \;:=\; U^*\left(\frac{|x-\xi|}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right).$$

In this definition, we have to introduce a small correction term a^ξ in order to assure that each droplet state u^ξ has exactly the same mass. Note that by Lemma 3.6 the correction a^ξ and its derivatives with respect to ξ_i, $i = 1, \dots, d$, are all exponentially small. In the absence of noise, the droplet state u^ξ is by construction an almost stationary solution to the Cahn–Hilliard equation. Let us show that this also holds true for the mass conserving Allen–Cahn equation.

Lemma 4.1 (Almost stationary solution to (SAC)**).**
Let $\xi \in \Omega_{\rho+\delta}$. We obtain

$$\varepsilon^2 \Delta u^\xi - f(u^\xi) + \frac{1}{|\Omega|}\int_\Omega f(u^\xi)\,\mathrm{d}x \;=\; \mathcal{O}(\exp) \;\; \text{in } \Omega, \quad \text{and} \quad \partial_\eta u^\xi \;=\; \mathcal{O}(\exp) \;\; \text{on } \partial\Omega.$$

Proof. First, we observe that Proposition 3.4 implies $\partial_\eta u^\xi = \mathcal{O}(\exp)$.
By the definition of u^ξ, we immediately see that $\varepsilon^2\Delta u^\xi - f(u^\xi) \;=\; \sigma(\frac{\rho - a^\xi}{\varepsilon})$. On the other hand, we have by Green's identity

$$\begin{aligned} \frac{1}{|\Omega|}\int_\Omega f(u^\xi)\,\mathrm{d}x \;&=\; \frac{1}{|\Omega|}\int_\Omega \varepsilon^2 \Delta u^\xi\,\mathrm{d}x - \sigma\left(\frac{\rho - a^\xi}{\varepsilon}\right) \\ &=\; \frac{\varepsilon^2}{|\Omega|}\int_{\partial\Omega} \frac{\partial u^\xi}{\partial\eta}\,\mathrm{d}S - \sigma\left(\frac{\rho - a^\xi}{\varepsilon}\right) \;=\; -\sigma\left(\frac{\rho - a^\xi}{\varepsilon}\right) + \mathcal{O}(\exp). \end{aligned}$$

Hence, the droplet state u^ξ is up to exponentially small terms a stationary solution to (SAC). $\square$

Further properties of the bubble u^ξ were collected in Proposition 3.4 and Lemma 3.5. By virtue of Lemma 4.1, the droplet fails to satisfy the boundary condition of (SAC) by an exponentially small term. In order to fix this, we have to add an exponentially small correction v^ξ. Here, we rely on the same correction term v^ξ that was used to fix the boundary condition for the Cahn–Hilliard equation in Theorem 3.7. In that case, the function $\tilde{u}^\xi := u^\xi + v^\xi$ also satisfies the boundary condition $\partial_\eta \Delta \tilde{u}^\xi = 0$, which is not necessary for our purpose. By a slight adaption of the proof of Theorem 3.7 (see [AF98], Theorem 5.1 for details), we expect that one could introduce a more fitting correction term. For convenience, we omit the technical details. Recall that v^ξ and its derivatives with respect to ξ are all exponentially small (cf. Proposition 3.7). Also, note that $\int_\Omega v^\xi(x)\,\mathrm{d}x = 0$ and thus, the mass is not influenced.

Similarly to Definition 3.8, provided the distance of the droplet's center ξ to the boundary of Ω is at least $\rho + \delta$ for some small $\delta > 0$, we define the set of all translates of the droplet $\tilde{u}^\xi := u^\xi + v^\xi$ as the slow manifold $\mathcal{M}_\rho$.

Definition 4.2 (The slow manifold)**.**
For a fixed radius $\rho > \bar{\rho} > 0$ and a minimal distance $\delta > 0$ of the droplets to the boundary $\partial\Omega$, we define the slow (C^3-) manifold

$$\mathcal{M}_\rho := \left\{ \tilde{u}^\xi := u^\xi + v^\xi \ : \ \xi \in \Omega_{\rho+\delta} \right\}.$$

Here, u^ξ denotes the droplet state constructed in Proposition 3.4 and v^ξ is an exponentially small correction given by Theorem 3.7. For $i, j \in \{1, \dots, d\}$, we denote the partial derivatives by $\tilde{u}_i^\xi = \partial_{\xi_i} \tilde{u}^\xi$, and $\tilde{u}_{ij}^\xi = \partial_{\xi_i}\partial_{\xi_j} \tilde{u}^\xi$, respectively. Also, note that $\mathcal{M}_\rho$ is nondegenerate and defines a d-dimensional manifold.

To conclude our collection of important results, we consider the eigenvalue problem for the mass conserving Allen–Cahn equation linearized at a droplet state $\tilde{u}^\xi \in \mathcal{M}_\rho$. Recall that it is crucial for the stability analysis that eigenvalues associated to eigenvectors orthogonal to the tangent space of $\mathcal{M}_\rho$ are negative and uniformly bounded away from zero. Moreover, the droplet is stable for the dynamics and the d exponentially small eigenvalues correspond to translations of $\tilde{u}^\xi$. We gave the main spectral result already in Theorem 3.12, but repeat it for completeness. Note that here the spectral gap is independent of the space dimension, as opposed to the linearized Cahn–Hilliard operator in H^{-1}.

Theorem 4.3 (The linearized Allen–Cahn operator, [ABF98], Proposition 2.2)**.**
Let $\tilde{u}^\xi \in \mathcal{M}_\rho$ and $\mu_1 \le \mu_2 \le \mu_3 \le \dots$ be the eigenvalues of

$$\begin{cases} \mathcal{L}^\xi \varphi \, = \, \varepsilon^2 \Delta\varphi - f'(\tilde{u}^\xi)\varphi + \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f'(\tilde{u}^\xi)\,\varphi\,\mathrm{d}x \, = \, -\mu\varphi, & x \in \Omega, \\ \partial_\eta\,\varphi \, = \, 0, & x \in \partial\Omega. \end{cases}$$

There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$

$$\mu_1, \dots, \mu_d \, = \, \mathcal{O}(\exp) \quad \text{and} \quad \mu_{d+1} \, > \, C\varepsilon^2.$$

The d-dimensional space W^ξ spanned by the eigenfunctions corresponding to the exponentially small eigenvalues can be represented by $W^\xi = \operatorname{span}\left\{ w_1^\xi, \dots, w_d^\xi \right\}$ and the normalized eigenfunctions w_i^ξ satisfy

$$\left\| w_i^\xi - \frac{u_i^\xi}{\left\| u_i^\xi \right\|} \right\|_{L^2} = \mathcal{O}(\exp) \quad \forall i \, = \, 1, \dots d. \tag{4.1}$$

4.2 The stochastic ODE for the droplet's motion

Here, we give the exact SDE on the slow manifold $\mathcal{M}_\rho$ and analyze it in terms of $\varepsilon > 0$ being small. Following Chapter 2, we first have to establish a coordinate system in a small tubular neighborhood of $\mathcal{M}_\rho$ (Definition 2.2), and then show the invertibility of the matrix $A(\xi, v)$ from Definition & Metatheorem 1. The next lemma deals with defining a pair of Fermi coordinates in a tubular neighborhood of $\mathcal{M}_\rho$. The coordinate system does not rely on the exact eigenfunctions w_i^ξ given by Theorem 4.3, but on the exact tangent space of the slow manifold $\mathcal{M}_\rho$ (cf. Remark 2.3).

Lemma 4.4 (Fermi coordinates, [ABF98], Lemma 2.4)**.**
Let $a > 0$ be a sufficiently small number. Then, the condition

$$\inf_{\xi \in \Omega_{\rho+2\delta}} \|u - \tilde{u}^\xi\|_{L^2} \;\leq\; a\varepsilon^{1/2}$$

implies the existence of a unique $\xi \in \Omega_{\rho+\delta}$ such that

$$\|u - \tilde{u}^\xi\|_{L^2} \;=\; \inf_{\zeta \in \Omega_{\rho+\delta}} \|u - \tilde{u}^\zeta\|_{L^2}.$$

Moreover, ξ is a smooth function of u and $\langle u - \tilde{u}^\xi, \tilde{u}_i^\xi\rangle = 0$ for $i \in \{1, \ldots, d\}$.

We analyze the matrix $A(\xi, v)$ from Definition & Metatheorem 1 and its inverse, which plays an important role in the derivation of the stochastic ODE governing the droplet's motion. In our case, the matrix $A(\xi, v)$ is defined by $A_{kj} \;=\; \langle \tilde{u}_k^\xi, \tilde{u}_j^\xi\rangle - \langle \tilde{u}_{kj}^\xi, v\rangle$.

Lemma 4.5 (Invertibility of the matrix $A(\xi, v)$)**.**
Let $\tilde{u}^\xi \in \mathcal{M}_\rho$. Then, we obtain

$$\langle \tilde{u}_k^\xi, \tilde{u}_j^\xi\rangle \;=\; \mathcal{X}\varepsilon^{-1}\delta_{kj} + \mathcal{O}(1),$$

where $\mathcal{X} = \int_{\mathbb{R}} U'(y)^2 \, \mathrm{d}y$ for U being the heteroclinic defined in Lemma 3.5.
Moreover, as long as $\|v\|_{L^2} < \varepsilon^{k-2}$ for some $k > 5/2$, the matrix A is invertible with

$$A(\xi, v)^{-1} \;=\; \mathcal{X}^{-1}\varepsilon \, \mathrm{I}_d + \mathcal{O}(\varepsilon^{k-3/2}).$$

Proof. The estimate of the inner product $\langle \tilde{u}_k^\xi, \tilde{u}_j^\xi\rangle$ follows from the definition of u^ξ, Proposition 3.4(v), and the exponential estimates of v^ξ and its derivatives (see also Lemma 3.33 or the proof of Lemma 4.8).
The analysis of the inverse matrix can be carried out similarly to the proof of Lemma 3.14 via an argument involving the geometric series. In more detail, with the L^2-estimates of Lemma 3.33 we obtain $|\langle \tilde{u}_{kj}^\xi, v\rangle| \leq c\varepsilon^{-3/2}\|v\|_{L^2}$ and thus,

$$A \;=\; \mathcal{X}\varepsilon^{-1}\left[\mathrm{I}_d - \mathcal{O}(\varepsilon^{-1/2}\|v\|)\right].$$

Since $\|v\|_{L^2} < \varepsilon^{k-2}$ for $k > 5/2$, we can invert the matrix A and obtain

$$A^{-1} \;=\; \mathcal{X}^{-1}\varepsilon\left[\mathrm{I}_d + \mathcal{O}(\varepsilon^{-1/2}\|v\|)\right] \;=\; \mathcal{X}^{-1}\varepsilon \, \mathrm{I}_d + \mathcal{O}(\varepsilon^{k-3/2}). \qquad \square$$

Finally, we give the rigorous dynamics governing the motion of the droplet's center ξ. Recall that by Theorem 2.6 and the justification in Lemma 2.8, the exact SDE in the new coordinate frame for the droplet's motion is given by the Itô diffusion

$$d\xi \;=\; b(\xi)\, dt + \langle \sigma(\xi), dW\rangle,$$

with

$$\sigma_r(\xi) = \sum_i A_{ri}^{-1} \tilde{u}_i^\xi \tag{4.2}$$

and

$$\begin{aligned} b_r(\xi) &= \sum_i A_{ri}^{-1} \langle \tilde{u}_i^\xi, \mathcal{A}(\tilde{u}^\xi + v)\rangle + \sum_i A_{ri}^{-1} \sum_j \langle \tilde{u}_{ij}^\xi, \mathcal{Q}\sigma_j\rangle \\ &\quad + \sum_{i,j,k} A_{ri}^{-1} \left[\frac{1}{2}\langle \tilde{u}_{ijk}^\xi, v\rangle - \langle \tilde{u}_{ij}^\xi, \tilde{u}_k^\xi\rangle - \frac{1}{2}\langle \tilde{u}_i^\xi, \tilde{u}_{jk}^\xi\rangle\right] \langle \mathcal{Q}\sigma_j, \sigma_k\rangle. \end{aligned} \tag{4.3}$$

Analysis of the SDE

In order to analyze the SDE in terms of $\varepsilon > 0$ and to show stochastic stability later in Section 4.3, we need to bound the diffusion (4.2) and the drift term (4.3). Since we cannot control the cubic nonlinearity in L^2, we need to assume additional smallness in H^1. In the following analysis, we consider the maximal radius such that the Fermi coordinates, and thus the stochastic ODE, are well defined. In our main stability result (Theorem 4.20), we have to consider a smaller neighborhood of the slow manifold.

Lemma 4.6. *Assume that $\xi \in \Omega_{\rho+\delta}$ and $\|v\|_{L^2} < \varepsilon^{k-2}$ for some $k > 5/2$. Then, we obtain*

$$\sigma_r(\xi) = \mathcal{X}^{-1}\varepsilon u_r^\xi + \mathcal{O}_{L^2}(\varepsilon^{k-2}).$$

Proof. This is a direct consequence of the definition (4.2) of σ and the bound on the inverse of the matrix A in Lemma 4.5. □

As a next step, let us split the diffusion process ξ into a purely deterministic part and remaining terms $\mathcal{A}(\xi, v)$, which only arise due to the presence of noise. We write

$$d\xi_r = \sum_i A_{ri}^{-1} \langle \tilde{u}_i^\xi, \mathcal{A}(\tilde{u}^\xi + v)\rangle\, dt + d\mathcal{A}_r(\xi, v),$$

where $\mathcal{A}_r$ can be computed easily from the expressions (4.2) and (4.3). Also, note that most of the terms in $\mathcal{A}(\xi, v)$ stem from Itô correction terms. The following lemma gives a bound on the deterministic part. This is the only term where we need to work in H^1.

Lemma 4.7. *Let $\xi \in \Omega_{\rho+\delta}$. Moreover, for some $k > 5/2$ and fixed small $\kappa > 0$ assume that $\|v\|_{L^2} < \varepsilon^{k-2}$ and $\|\nabla v\|_{L^2} < \varepsilon^{k-4-2\kappa/d}$. We have*

$$|\langle \mathcal{L}(\tilde{u}^\xi + v), \tilde{u}_i^\xi\rangle| = \mathcal{O}\left(\varepsilon^{\min\{k-5/2\,,\,3k-7-d-\kappa\}}\right).$$

Proof. By Lemma 4.1, we have $\mathcal{L}(\tilde{u}^\xi) = \mathcal{O}(\exp)$. For the nonlinear terms we obtain

$$\begin{aligned} \langle \mathcal{N}^\xi(v), \tilde{u}_i^\xi\rangle &= \int_\Omega (3\tilde{u}^\xi v^2 + v^3)\, \tilde{u}_i^\xi \,\mathrm{d}x \le C\|\tilde{u}_i^\xi\|_\infty \left[\|v\|_{L^2}^2 + \|v\|_{L^3}^3\right] \\ &\le C\varepsilon^{-1}\left[\|v\|_{L^2}^2 + \|v\|_{L^2}^{3-d/2}\|\nabla v\|_{L^2}^{d/2}\right] \le C\varepsilon^{2k-5} + C\varepsilon^{3k-7-d-\kappa}, \end{aligned}$$

where we utilized that $\|\tilde{u}_i^\xi\|_\infty = \mathcal{O}(\varepsilon^{-1})$ by Lemma 3.33 and interpolated the L^3-norm between L^2 and H^1 via Nirenberg's inequality. It remains to control the term involving the linearization

$$\begin{aligned} \langle \mathcal{L}^\xi v, \tilde{u}_i^\xi\rangle = \langle v, \mathcal{L}^\xi[\tilde{u}_i^\xi]\rangle &= \langle v, \varepsilon^2\Delta\tilde{u}_i^\xi - f'(\tilde{u}^\xi)\tilde{u}_i^\xi\rangle \\ &\le \varepsilon^2\langle v, \Delta\tilde{u}_i^\xi\rangle + C\|v\|_{L^2}\|\tilde{u}_i^\xi\|_{L^2} \le C\varepsilon^{k-5/2} + \varepsilon^{k-2}\|\tilde{u}_i^\xi\|_{L^2} \le C\varepsilon^{k-5/2}. \end{aligned}$$

Here, we used that $\mathcal{L}^\xi$ defines a selfadjoint operator on $L^2_0(\Omega)$ and that v is orthogonal to constants. Moreover, we utilized $\Delta\tilde{u}^\xi_i = \Delta_\xi u^\xi_i + \mathcal{O}(\exp)$ by definition of u^ξ and the exponential bounds on the correction terms a^ξ and v^ξ. This shows that $\|\Delta\tilde{u}^\xi_i\|_{L^2} = \mathcal{O}(\varepsilon^{-5/2})$ (see also Lemma 3.33). □

Before we analyze the terms in $\mathcal{A}_r$, we establish a better estimate of the scalar product $\langle\partial_\xi\tilde{u}^\xi, \partial^2_\xi\tilde{u}^\xi\rangle$ than with the Cauchy–Schwarz inequality, which would yield $\mathcal{O}(\varepsilon^{-2})$. We show that the inner product is even exponentially small.

Lemma 4.8. *Let $\xi \in \Omega_{\rho+\delta}$ and $i, j, k \in \{1, \dots, d\}$. The following estimate holds true*

$$|\langle\tilde{u}^\xi_{ij}, \tilde{u}^\xi_k\rangle| = \mathcal{O}(\exp).$$

Proof. For simplicity of presentation, we present only the case $d = 2$ and $i = j = k$.
The other cases work essentially in the same way. One always ends up with the product of a radial function with odd powers of sine and cosine. In spherical coordinates though, the number of different cases is tedious and we omit the details.

Since ξ lies in $\Omega_{\rho+\delta}$, the ball $B_{\rho+\delta/2}(\xi)$ is completely contained in Ω. By Proposition 3.4(v), we know that $\tilde{u}^\xi_k, \tilde{u}^\xi_{kk} = \mathcal{O}(\exp)$ on $\Omega \setminus B_{\rho+\delta/2}(\xi)$. Therefore, it remains to compute the inner product on the ball $B_{\rho+\delta/2}(\xi)$. Recall that we defined $u^\xi(x) = U^*(r/\varepsilon, (\rho - a^\xi)/\varepsilon)$, where we set $r := |x - \xi|$. Differentiating u^ξ with respect to ξ_k yields

$$u^\xi_k(x) = \varepsilon^{-1}U^*_1\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)\frac{\partial r}{\partial\xi_k} + \mathcal{O}(\exp),$$

where we utilized that the partial derivatives of the correction term a^ξ are exponentially small (cf. Lemma 3.6). Similarly, we obtain for the second derivative

$$\begin{aligned} u^\xi_{kk}(x) &= \varepsilon^{-2}\,U^*_{11}\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)\left(\frac{\partial r}{\partial\xi_k}\right)^2 + \varepsilon^{-1}\,U^*_1\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)\frac{\partial^2 r}{\partial^2\xi_k} + \mathcal{O}(\exp) \\ &= \varepsilon^{-2}\,U^*_{11}\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)\left(\frac{\partial r}{\partial\xi_k}\right)^2 - \varepsilon^{-1}\,U^*_1\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)\frac{1}{r}\left(1 + \left(\frac{\partial r}{\partial\xi_k}\right)^2\right) + \mathcal{O}(\exp). \end{aligned}$$

We define $\mathrm{cs}(\varphi) = \cos(\varphi)$ for $k = 1$, and $\mathrm{cs}(\varphi) = \sin(\varphi)$ for $k = 2$. By a transformation to polar coordinates, we obtain

$$\begin{aligned} &\int_{B_{\rho+\delta/2}(\xi)} U^*_1\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)U^*_{11}\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)\left(\frac{\partial r}{\partial\xi_k}\right)^3 \mathrm{d}x \\ &= \int_0^{2\pi}\int_0^{\rho+\delta/2} U^*_1\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)U^*_{11}\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)\mathrm{cs}^3(\varphi)\,\mathrm{d}r\,\mathrm{d}\varphi = 0, \end{aligned}$$

since $\int_0^{2\pi}\mathrm{cs}^3(\varphi)\,\mathrm{d}\varphi = 0$. Similarly, one computes

$$\begin{aligned} &\int_{B_{\rho+\delta/2}(\xi)} \frac{1}{r}U^*_1\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)^2\left(\left(\frac{\partial r}{\partial\xi_k}\right) + \left(\frac{\partial r}{\partial\xi_k}\right)^3\right)\mathrm{d}x \\ &= \int_0^{2\pi}\int_0^{\rho+\delta/2} U^*_1\left(\frac{r}{\varepsilon}, \frac{\rho - a^\xi}{\varepsilon}\right)^2\left(\mathrm{cs}(\varphi) + \mathrm{cs}^3(\varphi)\right)\mathrm{d}r\,\mathrm{d}\varphi = 0. \end{aligned}$$

Combined we derived

$$\langle u^\xi_{kk}, u^\xi_k\rangle_{L^2(\Omega)} = \langle u^\xi_{kk}, u^\xi_k\rangle_{L^2(\Omega\setminus B_{\rho+\delta/2}(\xi))} + \mathcal{O}(\exp) = \mathcal{O}(\exp).$$

Note that we neglected the correction term v^ξ in our calculations. Since v^ξ and all its derivatives with respect to ξ are exponentially small, the effect of this additional term is hidden in the $\mathcal{O}(\exp)$-term. $\square$

With this estimate at hand, we can finally analyze the term $d\mathcal{A}_r(\xi, v)$. Here, it is sufficient to control v in L^2, but since $\mathcal{A}$ collects all the terms stemming from stochastics, the estimate will obviously depend on the noise strength η_0 (cf. Assumption 4.1).

Lemma 4.9. *Assume that $\xi \in \Omega_{\rho+\delta}$ and $\|v\|_{L^2} < \varepsilon^{k-2}$ for $k > 5/2$. Then, we obtain*

$$d\mathcal{A}_r(\xi, v) = \mathcal{X}^{-2}\varepsilon^2 \langle \tilde{u}^\xi_{rr}, \mathcal{Q} u^\xi_r \rangle \, dt + \mathcal{O}(\varepsilon^{k-5/2}\eta_0)\, dt + \langle \mathcal{X}^{-1}\varepsilon\, \tilde{u}^\xi_r + \mathcal{O}_{L^2}(\varepsilon^{k-3/2}), dW \rangle .$$

Proof. With (4.2) and (4.3) we see that

$$\begin{aligned} d\mathcal{A}_r &= \sum_i A^{-1}_{ri} \sum_j \langle \tilde{u}^\xi_{ij}, \mathcal{Q}\sigma_j \rangle \, dt + \sum_{i,j,k} A^{-1}_{ri} \left[\frac{1}{2} \langle \tilde{u}^\xi_{ijk}, v \rangle - \langle \tilde{u}^\xi_{ij}, \tilde{u}^\xi_k \rangle - \frac{1}{2} \langle \tilde{u}^\xi_i, \tilde{u}^\xi_{jk} \rangle \right] \langle \mathcal{Q}\sigma_j, \sigma_k \rangle \, dt \\ &\quad + \sum_i A^{-1}_{ri} \langle \tilde{u}^\xi_i, dW \rangle . \end{aligned}$$

The claim follows now directly from the estimates of Lemmata 4.5, 4.6, and 4.8. $\square$

In order to approximate the full effective dynamics of the droplet's center ξ, we combine the estimate of $d\mathcal{A}_r$ with the estimate of Lemma 4.7.

Theorem 4.10 (Approximation of the effective dynamics)**.**
Let $\xi \in \Omega_{\rho+\delta}$. As long as $\|v\|_{L^2} < \varepsilon^{k-2}$ and $\|\nabla v\|_{L^2} < \varepsilon^{k-4-2\kappa/d}$ for some $k > 5/2$ and some small $\kappa > 0$, we have

$$\begin{aligned} d\xi_r &= \mathcal{X}^{-2}\varepsilon^2 \langle \tilde{u}^\xi_{rr}, \mathcal{Q}\tilde{u}^\xi_r \rangle \, dt + \mathcal{X}^{-1}\varepsilon \, \langle \tilde{u}^\xi_r, dW \rangle \\ &\quad + \mathcal{O}(\varepsilon^{k-5/2}\eta_0 + \varepsilon^{\min\{k-1/2\,,\,3k-5-d-\kappa\}})\, dt + \langle \mathcal{O}_{L^2}(\varepsilon^{k-3/2}), dW \rangle , \end{aligned}$$

where $\mathcal{X} = \int_{\mathbb{R}} U'(y)^2 \, dy$ for U being the heteroclinic defined in Lemma 3.5.

Remark 4.11. Analogously to Lemma 3.18, we see that the motion of the droplet is in first approximation given by the projection of the Wiener process onto the tangent space of $\mathcal{M}_\rho$ at $\tilde{u}^\xi$, plus a small error term dR given by Theorem 4.10, i.e., we obtain

$$d\xi_r = \sum_i S^{-1}_{ri} \langle \tilde{u}^\xi_i, \circ \, dW \rangle + dR,$$

where the matrix S corresponds to the first fundamental form of the manifold $\mathcal{M}_\rho$ and is given by $S_{ij} = \langle \tilde{u}^\xi_i, \tilde{u}^\xi_j \rangle$.

4.3 Stochastic Stability

In this section, we investigate the stochastic stability with respect to various norms. Here, we follow the method introduced in Section 2.3 very closely. In order to be able to define the Fermi coordinates (cf. Lemma 4.4) and thus to establish the effective dynamics, we first need to show that the L^2-norm of v stays sufficiently small for very long times. For controlling the nonlinear terms in the stochastic ODE, we extend the stability result afterwards to the space H^1_ε, a weighted Sobolev space that was also used in [ABBK15] and already introduced in the stability analysis for the stochastic Cahn–Hilliard equation in Section 3.3.

4.3.1 Stability in L^2

Recall that the normal component $v = u - \tilde{u}^\xi$ satisfies by (2.13)

$$dv = \Big[\mathcal{O}(\exp) + \mathcal{L}^\xi v + \mathcal{N}^\xi(v)\Big]\, dt + dW - \sum_i \tilde{u}_i^\xi \, d\xi_i - \frac{1}{2}\sum_{i,j} \tilde{u}_{ij}^\xi \langle \mathcal{Q}\sigma_i, \sigma_j\rangle \, dt.$$

Using the spectral estimates of Theorem 4.3, we can bound the quadratic form orthogonal to the tangent space of $\mathcal{M}_\rho$ spanned by the derivatives $\tilde{u}_i^\xi$ (see Assumption 2.1).

Lemma 4.12. *Suppose that $v \perp \tilde{u}_i^\xi$ for $i = 1, \dots, d$. Then, we obtain*

$$\langle \mathcal{L}^\xi v, v\rangle \le -C\varepsilon^2 \|v\|_{L^2}^2 + \mathcal{O}(\exp).$$

Proof. The statement follows directly from the spectral estimates of Theorem 4.3 combined with Theorem 2.10, since $\langle v, w_i\rangle = \langle v, w_i - \frac{\tilde{u}_i^\xi}{\|\tilde{u}_i^\xi\|}\rangle = \mathcal{O}(\exp)\|v\|$. □

Recall (3.2), where for $f \in H^1(\Omega)$ we defined the weighted norm

$$\|f\|_{H^1_\varepsilon} := \Big(\varepsilon^2 \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2\Big)^{1/2}. \tag{4.4}$$

Note that H^1_ε is equivalent to the Sobolev space H^1 and we always have $\|f\|_{H^1} \le \varepsilon^{-1}\|f\|_{H^1_\varepsilon}$. As long as $\|v\|_{L^2}$ stays sufficiently small, we can give the main estimate for showing stability (Metatheorem 2), namely a negative upper bound on $\langle \mathcal{L}(u^\xi + v), v\rangle$. Compared to the analysis of the stochastic ODE governing the droplet's motion in Section 4.2, we have to work in a smaller neighborhood of the slow manifold $\mathcal{M}_\rho$.

Theorem 4.13. *As long as $\xi \in \Omega_{\rho+\delta}$ and $\|v\|_{L^2} < \varepsilon^{k-2}$ for $k > 4 + d/2$, we obtain for some constant $c > 0$*

$$\langle \mathcal{L}(\tilde{u}^\xi + v), v\rangle \le -c\varepsilon^2 \|v\|_{L^2}^2.$$

Proof. In the two-dimensional case, we follow [ABBK15, p.15]. First, we extend the estimate of Lemma 4.12 to H^1_ε. For $\gamma \in (0,1)$, we obtain

$$\langle \mathcal{L}^\xi v, v\rangle \le -\gamma\varepsilon^2 \|\nabla v\|_{L^2}^2 + \gamma \|f'(u^\xi)\|_\infty \|v\|_{L^2}^2 - C(1-\gamma)\varepsilon^2 \|v\|_{L^2}^2.$$

Note that the extra term vanishes as v is orthogonal to constants. We use now that $f'(u^\xi)$ is uniformly bounded and fix $\gamma = \varepsilon^2$. This yields

$$\langle \mathcal{L}^\xi v, v\rangle \le -c_0\varepsilon^2 \|v\|_{H^1_\varepsilon}^2.$$

For the nonlinear terms, we obtain via the Gagliardo–Nirenberg inequality

$$\langle \mathcal{N}^\xi(v), v\rangle \le C\|v\|_{L^3}^3 \le C\|v\|_{H^1}\|v\|_{L^2}^2 \le C\varepsilon^{-1}\|v\|_{H^1_\varepsilon}\|v\|_{L^2}^2.$$

As long as $\|v\|_{L^2} < c\varepsilon^3$, we have

$$\langle \mathcal{L}(u^\xi + v), v\rangle \le -C\varepsilon^2 \|v\|_{H^1_\varepsilon}^2 \le -C\varepsilon^2 \|v\|_{L^2}^2.$$

In the three-dimensional case, we obtain by Gagliardo–Nirenberg and Young's inequality

$$\begin{aligned}\langle \mathcal{N}^\xi(v), v\rangle &\le C\|v\|_{L^3} \le C\|v\|_{H^1}^{3/2}\|v\|_{L^2}^{3/2} \le c\varepsilon^2\|v\|_{H^1_\varepsilon}^2 + c\varepsilon^{-12}\|v\|_{L^2}^6 \\ &\le c\varepsilon^2\|v\|_{H^1_\varepsilon}^2 + c\varepsilon^2\|v\|_{L^2}^2 \le C\varepsilon^2\|v\|_{H^1_\varepsilon}^2,\end{aligned}$$

where we used that by assumption $\|v\|_{L^2} < c\varepsilon^{7/2}$. □

By Metatheorems 3 and 4, it remains to control the remaining terms for $d\|v\|^2$. These are of order $\mathcal{O}(\eta_0)$: Lemmata 4.5 and 4.6 imply that $\|\sigma\|_{L^2} = \mathcal{O}(\varepsilon^{1/2})$. By using the estimates of Lemma 3.33, we obtain with the Cauchy–Schwarz inequality

$$\begin{aligned}\langle dv, dv\rangle &= \operatorname{trace}(\mathcal{Q})\,dt + \sum_{i,j}\langle \tilde{u}_i^\xi, \tilde{u}_j^\xi\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\,dt - 2\sum_i \langle \tilde{u}_i^\xi, \mathcal{Q}\sigma_i\rangle\,dt \\ &\le \Big(\eta_0 + \mathcal{X}\varepsilon^{-1}C\eta_1\varepsilon^{1/2}\varepsilon^{1/2} + C\varepsilon^{-1/2}\eta_1\varepsilon^{1/2}\Big)\,dt \,=\, \mathcal{O}(\eta_0)\,dt.\end{aligned}$$

and

$$\frac{1}{2}\sum_{i,j}\langle \tilde{u}_{ij}^\xi, v\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\,dt \,\le\, C\varepsilon^{-3/2}\|v\|\eta_1\varepsilon^{1/2}\varepsilon^{1/2}\,dt \,\le\, C\varepsilon^{k-5/2}\eta_0\,dt.$$

We summarize all the previous estimates in the following theorem, which is of the same type as Theorem 2.13 in the general framework.

Theorem 4.14. *As long as $\xi \in \Omega_{\rho+\delta}$ and $\|v\|_{L^2} < c\varepsilon^{k-2}$ for some $k > 4 + d/2$, we obtain*

$$d\|v\|^2 \,\le\, \Big[-C\varepsilon^2\|v\|^2_{L^2} + C\eta_0\Big]\,dt + 2\langle v, dW\rangle.$$

With this stochastic differential inequality at hand, we can finally show that solutions to the mass conserving stochastic Allen–Cahn equation (SAC) stay close (in L^2) to the slow manifold $\mathcal{M}_\rho$ for very long times with high probability.

Theorem 4.15 (L^2-stability)**.**
Let $d \in \{2, 3\}$ and $k > 4 + d/2$. For a solution $u = \tilde{u}^\xi + v$ to (SAC) with $\xi \in \Omega_{\rho+\delta}$ and $v \perp \tilde{u}_j^\xi$ for all $j \in \{1, \dots, d\}$ consider the exit time

$$\tau^* \,=\, \inf\Big\{t \in [0, T_\varepsilon \wedge \tau_0] \,:\, \|v(t)\| \,>\, \varepsilon^{k-2}\Big\},$$

with a deterministic cut-off $T_\varepsilon = \varepsilon^{-N}$ for any fixed large $N > 0$ and τ_0 the exit time from $\Omega_{\rho+\delta}$. Furthermore, suppose that for some $0 < \nu < 1$ and some $\kappa > 0$ very small

$$\|v(0)\| \,\le\, \nu\varepsilon^{k-2} \quad \text{and} \quad \eta_0 \,\le\, C\varepsilon^{2k-2+\kappa}.$$

Then, the probability $\mathbb{P}(\tau^* \wedge \tau_0 < T_\varepsilon)$ is smaller than any power of ε, as ε tends to 0. And thus, for very large time scales the solution stays close to the slow manifold $\mathcal{M}_\rho$ with high probability.

Proof. The statement follows directly from Theorem 4.14 and the general stability result of Theorem 2.14. □

4.3.2 Stability in H^1_ε

We use the long-time stability in L^2 to extend the stability result to H^1_ε defined by (4.4). Since Theorem 4.15 provides us with a good control of $\|v\|$, it remains to show that $\|\nabla v\|$ stays small for long times. For this purpose, we consider the following relation

$$d\|\nabla v\|^2 \,=\, 2\langle \nabla v, d\nabla v\rangle + \langle \nabla dv, \nabla dv\rangle \,=\, -2\langle \Delta v, dv\rangle + \langle \nabla dv, \nabla dv\rangle,$$

where we integrated once by part as v satisfies Neumann boundary conditions. By series expansion of W we obtain

$$\begin{aligned}\langle \nabla\tilde{u}_i^\xi, \nabla dW\rangle\langle \sigma_i, dW\rangle &= \sum_{k\in\mathbb{N}}\alpha_k^2\langle \nabla\tilde{u}_i^\xi, \nabla e_k\rangle\langle \sigma_i, e_k\rangle\,dt \,\le\, \sum_{k\in\mathbb{N}}\alpha_k\|\nabla e_k\|\alpha_k\|\nabla\tilde{u}_i^\xi\|\|\sigma_i\|\,dt \\ &\le \eta_2^{1/2}\eta_0^{1/2}\|\nabla\tilde{u}_i^\xi\|\|\sigma_i\|\,dt \,\le\, C\eta_2^{1/2}\eta_0^{1/2}\varepsilon^{-3/2}\varepsilon^{1/2}\,dt \le C(\varepsilon^{-2}\eta_0 + \eta_2)\,dt.\end{aligned}$$

This yields

$$\begin{aligned}\langle\nabla dv,\nabla dv\rangle \;&=\; \langle\nabla dW,\nabla dW\rangle - 2\sum_i\langle\nabla\tilde u_i^\xi,\nabla dW\rangle\langle\sigma_i,dW\rangle + \sum_{i,j}\langle\nabla\tilde u_i^\xi,\nabla\tilde u_j^\xi\rangle\langle\mathcal{Q}\sigma_i,\sigma_j\rangle\,dt\\ &\le\; \Big[\eta_2 + C(\varepsilon^{-2}\eta_0+\eta_2) + C\varepsilon^{-3/2}\varepsilon^{-3/2}\eta_1\varepsilon^{1/2}\varepsilon^{1/2}\Big]\,dt \;=\; \mathcal{O}(\varepsilon^{-2}\eta_0+\eta_2)\,dt.\end{aligned}\tag{4.5}$$

Next, we consider the term $-\langle\Delta v, dv\rangle$. With $\langle\Delta v,1\rangle=0$ we obtain

$$\begin{aligned}-\langle\Delta v,dv\rangle \;=\;& -\langle\Delta v,\varepsilon^2\Delta v\rangle\,dt - \langle\Delta v, f(\tilde u^\xi)-f(\tilde u^\xi+v)\rangle\,dt - \langle\Delta v, dW - \sum_i\tilde u_i^\xi\langle\sigma_i,dW\rangle\rangle\\ &+\langle\Delta v,\sum_i\tilde u_i^\xi\,b_i(\xi)+\tfrac12\sum_{i,j}\tilde u_{ij}^\xi\langle\mathcal{Q}\sigma_i,\sigma_j\rangle+\mathcal{O}(\exp)\rangle\,dt \;=:\; T_1+T_2+T_3+T_4.\end{aligned}$$

We easily see that $T_1=-\varepsilon^2\|\Delta v\|^2\,dt$, which is a good term for our analysis. For the martingale term T_3 we derive

$$\begin{aligned}T_3 \;&=\; \sum_i\langle\Delta v,\tilde u_i^\xi\rangle\langle\sigma_i,dW\rangle - \langle\Delta v,dW\rangle \;=\; -\sum_i\langle\nabla v,\nabla\tilde u_i^\xi\rangle\langle\sigma_i,dW\rangle + \langle\nabla v,d\nabla W\rangle\\ &=\; \langle\mathcal{O}_{L^2}(\varepsilon^{-1}\|\nabla v\|),dW\rangle + \langle\mathcal{O}_{L^2}(\|\nabla v\|),dW\rangle.\end{aligned}$$

The term T_4 involves the drift term $b(\xi)$, which by Theorem 4.10 can only be bounded up to a stopping time.

Definition 4.16. For some given large deterministic $T_\varepsilon>0$, small $\kappa>0$, and $k>4+d/2$ we define the stopping time

$$\tau_\varepsilon \;:=\; \inf\Big\{t\in[0,T_\varepsilon\wedge\tau_0]\;:\;\|v\|_{L^2}\;>\;\varepsilon^{k-2}\quad\text{or}\quad\|\nabla v\|_{L^2}\;>\;\varepsilon^{k-4-2\kappa/d}\Big\}.$$

Theorem 4.10 implies that up to the stopping time τ_ε the drift term b is uniformly bounded by $\sup_{0\le t\le\tau_\varepsilon}|b(\xi)|\;\le\;c\eta_0+c\varepsilon^{k-1/2}$. This yields for $t\le\tau_\varepsilon$

$$T_4\;=\;\mathcal{O}\Big(\Big[\varepsilon^{-1/2}\eta_0+\varepsilon^{k-1}\Big]\,\|\Delta v\|\Big).$$

It remains to control the term T_2 involving the nonlinearity. First, let us note that integration by parts yields $\|\nabla v\|_{L^2}^2=-\langle\Delta v,v\rangle\le\|\Delta v\|_{L^2}\|v\|_{L^2}$, and that $\|v\|_{H^1}\le c\|\nabla v\|_{L^2}$ by Poincaré's inequality, since v has mean zero. Moreover, Nirenberg's inequality gives

$$\|v\|_{L^4}\;\le\;C\|v\|_{L^2}^{1-d/4}\|\nabla v\|_{L^2}^{d/4}.$$

By using the L^2-bound from Theorem 4.15, we obtain for $t\le\tau_\varepsilon$

$$\begin{aligned}T_2 &= \langle\Delta v,-v+3(\tilde u^\xi)^2v+3\tilde u^\xi v^2+v^3\rangle\,dt = \Big(\|\nabla v\|^2-\int_\Omega\nabla(3(\tilde u^\xi)^2v+3\tilde u^\xi v^2+v^3)\nabla\tilde u^\xi\,\mathrm{d}x\Big)\,dt\\ &= \Big(\|\nabla v\|^2-\int_\Omega(3(\tilde u^\xi)^2+6\tilde u^\xi v+3v^2)|\nabla v|^2\,\mathrm{d}x-\int_\Omega(6\tilde u^\xi v+3v^2)(\nabla\tilde u^\xi\nabla v)\,\mathrm{d}x\Big)\,dt\\ &\le \Big(\|\nabla v\|^2+c\varepsilon^{-1}\|v\|_{L^4}^2\|\nabla v\|+c\varepsilon^{-1}\|v\|\|\nabla v\|\Big)\,dt\\ &\le \Big(\|\nabla v\|^2+c\varepsilon^{-1}\|v\|^{2-d/2}\|v\|_{H^1}^{d/2}\|\nabla v\|+c\varepsilon^{-1}\|v\|\|\nabla v\|\Big)\,dt\\ &\le \Big(\|\nabla v\|^2+c\varepsilon^{-1}\|v\|^{2-d/2}\|\nabla v\|^{1+d/2}+c\varepsilon^{-1}\|v\|\|\nabla v\|\Big)\,dt\\ &\le \Big(\|v\|\|\Delta v\|+c\varepsilon^{-1}\|v\|^{5/2-d/4}\|\Delta v\|^{1/2+d/4}+c\varepsilon^{-1}\|v\|^{3/2}\|\Delta v\|^{1/2}\Big)\,dt\\ &\le \Big(\frac12\varepsilon^2\|\Delta v\|^2+c\varepsilon^{-2}\|v\|^2+c\varepsilon^{-\frac{12+2d}{6-d}}\|v\|^{\frac{20-2d}{6-d}}\Big)\,dt\\ &\le \Big(\frac12\varepsilon^2\|\Delta v\|^2+c\varepsilon^{2k-6}+c\varepsilon^{\frac{-52+2d+20k-2dk}{6-d}}\Big)\,dt.\end{aligned}$$

We observe that $\frac{-52+2d+20k-2dk}{6-d} = 2k-6+4\frac{2(k-2)+d}{6-d} > 2k-6$ and thus

$$T_2 \le \left(\frac{1}{2}\varepsilon^2\|\Delta v\|^2 + c\varepsilon^{2k-6}\right) dt.$$

Combining the estimates of $T_1, \dots, T_4$ with (4.5), we derive

$$\begin{aligned} d\|\nabla v\|^2 \;=\; & -\varepsilon^2\|\Delta v\|^2\,dt + \mathcal{O}(\varepsilon^{2k-6} + \varepsilon^{-3}\eta_0^2 + \varepsilon^{-2}\eta_0 + \eta_2)\,dt \\ & + \langle \mathcal{O}(\varepsilon^{-1}\|\nabla v\|), dW\rangle + \langle \mathcal{O}(\|\nabla v\|), dW\rangle, \end{aligned} \tag{4.6}$$

where we utilized Young's inequality for T_3. For the final step, we use that by an argument based on Poincaré's inequality $\|\nabla v\| \le c_0\|\Delta v\|$, and derive the following estimate of $d\|\nabla v\|^2$.

Lemma 4.17. *If $k > 4+d/2$ and $t \le \tau_\varepsilon$, with τ_ε given by Definition 4.16, the following relation holds true:*

$$d\|\nabla v\|^2 + c_0\varepsilon^2\|\nabla v\|^2\,dt \;=\; \Gamma_\varepsilon\,dt + \langle Z_\varepsilon, dW\rangle + \langle \Psi_\varepsilon, d\nabla W\rangle, \tag{4.7}$$

where

$$\Gamma_\varepsilon \;=\; \mathcal{O}(\varepsilon^{2k-6} + \varepsilon^{-3}\eta_0^2 + \varepsilon^{-2}\eta_0 + \eta_2) \tag{4.8}$$

and

$$\|Z_\varepsilon\|_{L^2}^2 \;=\; \mathcal{O}(\varepsilon^{-2}\|\nabla v\|^2), \quad \|\Psi_\varepsilon\|_{L^2}^2 \;=\; \mathcal{O}(\|\nabla v\|^2). \tag{4.9}$$

For establishing long-time stability in H^1, we follow Section 3.4 of [ABBK15]. Under the assumptions of Lemma 4.17, we estimate for any $p > 2$ the p-th moment of $\|\nabla v\|_{L^2}^2$ (see also the proof of Theorem 2.14). By Itô calculus we obtain

$$d\|\nabla v\|_{L^2}^{2p} \;=\; p\|v\|_{L^2}^{2p-2}\,d\|\nabla v\|_{L^2}^2 + p(p-1)\|\nabla v\|_{L^2}^{2p-4}\left[d\|\nabla v\|_{L^2}^2\right]^2.$$

Let us briefly comment on the estimate of the Itô correction term. Relation (4.7) implies that

$$\left[d\|\nabla v\|_{L^2}^2\right]^2 \;=\; \langle Z_\varepsilon, \mathcal{Q}Z_\varepsilon\rangle\,dt + \langle \Psi_\varepsilon, \Delta\mathcal{Q}\Psi_\varepsilon\rangle\,dt + 2\langle Z_\varepsilon, dW\rangle\langle \Psi_\varepsilon, d\nabla W\rangle, \tag{4.10}$$

and by series expansion, we see that

$$\begin{aligned} \langle Z_\varepsilon, dW\rangle\langle \Psi_\varepsilon, d\nabla W\rangle \;&=\; \sum_{k\in\mathbb{N}} \alpha_k^2 \langle Z_\varepsilon, e_k\rangle\langle \Psi_\varepsilon, \nabla e_k\rangle\,dt \\ \le \sum_{k\in\mathbb{N}} \alpha_k^2\|e_k\|\|\nabla e_k\|\|Z_\varepsilon\|\|\Psi_\varepsilon\|\,dt \;&\le\; \|Z_\varepsilon\|\|\Psi_\varepsilon\|\sqrt{\eta_0\eta_2} \;\le\; \|Z_\varepsilon\|^2\eta_0 + \|\Psi_\varepsilon\|^2\eta_2. \end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality, we derive

$$\left[d\|\nabla v\|_{L^2}^2\right]^2 \;\le\; C\left[\|Z_\varepsilon\|^2\eta_0 + \|\Psi_\varepsilon\|^2\eta_2\right]\,dt. \tag{4.11}$$

By plugging (4.7), (4.11), and (4.9) into the relation (4.10), we derive the following lemma by integrating.

Lemma 4.18. *Under the assumptions of Lemma 4.17, the following estimate holds true for any $p > 1$*

$$\mathbb{E}\|\nabla v(\tau_\varepsilon)\|_{L^2}^{2p} + cp\varepsilon^2 A_p \;\le\; \|\nabla v(0)\|_{L^2}^{2p} + C\left[\Gamma_\varepsilon + \varepsilon^{-2}\eta_0 + \eta_2\right] A_{p-1},$$

where the stopping time τ_ε is given by Definition 4.16 and A_p is defined as

$$A_p \;:=\; \mathbb{E}\int_0^{\tau_\varepsilon} \|\nabla v(s)\|_{L^2}^{2p}\,\mathrm{d}s.$$

For the sake of simplicity, we define $a_\varepsilon := C\varepsilon^{-2}\left[\Gamma_\varepsilon + \varepsilon^{-2}\eta_0 + \eta_2\right]$ and assume that the noise strength is small enough such that $a_\varepsilon < 1$. Note that by the definition of Γ_ε we thus also need $C\varepsilon^{2k-8} < 1$, which is true by assumption as $k > 4 + d/2$. Applying Lemma 4.18, we obtain inductively

$$
\begin{aligned}
A_p &\le C\varepsilon^{-2}\|\nabla v(0)\|_{L^2}^{2p} + Ca_\varepsilon A_{p-1} \le C\varepsilon^{-2}\|\nabla v(0)\|_{L^2}^{2p} + Ca_\varepsilon\varepsilon^{-2}\|\nabla v(0)\|_{L^2}^{2p-2} + a_\varepsilon^2 A_{p-2} \\
&\qquad\qquad \le \ldots \le C\varepsilon^{-2}\sum_{i=2}^{p} a_\varepsilon^{p-i}\|\nabla v(0)\|_{L^2}^{2i} + Ca_\varepsilon^{p-1}A_1.
\end{aligned}
$$

Note that Lemma 4.17 implies for $t \le \tau_\varepsilon$

$$
\mathbb{E}\int_0^t \|\nabla v(s)\|_{L^2}^2\,\mathrm{d}s \le C\varepsilon^{-2}\Gamma_\varepsilon T_\varepsilon + \varepsilon^{-2}\|\nabla v(0)\|_{L^2}^2 \le a_\varepsilon T_\varepsilon + \varepsilon^{-2}\|\nabla v(0)\|_{L^2}^2.
$$

Therefore, if we assume that $\|\nabla v(0)\|_{L^2}^2 < a_\varepsilon$, we derive for a constant C depending on p

$$
A_p \le C\varepsilon^{-2}\sum_{i=1}^{p} a_\varepsilon^{p-i}\|\nabla v(0)\|_{L^2}^{2i} + Ca_\varepsilon^p T_\varepsilon \le C\left[\varepsilon^{-2} + T_\varepsilon\right]a_\varepsilon^p. \tag{4.12}
$$

Lemma 4.19. *Let $k > 4 + d/2$ and τ_ε be as defined in Definition 4.16. Also, assume that*

$$
\Gamma_\varepsilon + \varepsilon^{-2}\eta_0 + \eta_2 \le C\varepsilon^{2k-6} \quad \text{and} \quad \|\nabla v(0)\|_{L^2}^2 \le a_\varepsilon < 1.
$$

Then, for any $p > 1$ it holds true that

$$
\mathbb{E}\|\nabla v(\tau_\varepsilon)\|_{L^2}^{2p} \le C\varepsilon^2\left[\varepsilon^{-2} + T_\varepsilon\right]a_\varepsilon^p.
$$

Proof. Lemma 4.18 and (4.12) imply

$$
\begin{aligned}
\mathbb{E}\|\nabla v(\tau_\varepsilon)\|_{L^2}^{2p} &\le \|\nabla v(0)\|_{L^2}^{2p} + C\left[\Gamma_\varepsilon + \varepsilon^{-2}\eta_0 + \eta_2\right]A_{p-1} \\
&= \|\nabla v(0)\|_{L^2}^{2p} + C\varepsilon^2 a_\varepsilon A_{p-1} \\
&\le \|\nabla v(0)\|_{L^2}^{2p} + C\varepsilon^2 a_\varepsilon\left[\varepsilon^{-2} + T_\varepsilon\right]a_\varepsilon^{p-1} \\
&\le Ca_\varepsilon^p + C\varepsilon^2 T_\varepsilon a_\varepsilon^p.
\end{aligned}
$$
□

With the help of Lemma 4.19, we can finally prove stability.

Theorem 4.20 (H^1-Stability)**.**
For $k > 4 + d/2$ and some small $\kappa > 0$, consider the exit time τ_ε given by Definition 4.16, i.e.,

$$
\tau_\varepsilon = \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0] : \|v(t)\|_{L^2} > \varepsilon^{k-2} \quad \text{or} \quad \|\nabla v(t)\|_{L^2} > \varepsilon^{k-4-2\kappa/d}\right\},
$$

where $T_\varepsilon = \varepsilon^{-N}$ for fixed large $N > 0$. Also, suppose that for some $\nu \in (0,1)$

$$
\|v(0)\|_{L^2} \le \nu\varepsilon^{k-2} \quad \text{and} \quad \|\nabla v(0)\|_{L^2} \le \nu\varepsilon^{k-4}.
$$

In addition, assume for the squared noise strength that

$$
\eta_0 \le C\varepsilon^{2k-2+\kappa} \quad \text{and} \quad \eta_2 \le C\varepsilon^{2k-6+\kappa}.
$$

Then, the probability $\mathbb{P}(\tau_\varepsilon < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as ε tends to 0.

Proof. We have

$$
\mathbb{P}(\tau_\varepsilon < T_\varepsilon \wedge \tau_0) \le \mathbb{P}\left(\|v(\tau_\varepsilon)\|_{L^2} > \varepsilon^{k-2}\right) + \mathbb{P}\left(\|\nabla v(\tau_\varepsilon)\|_{L^2} > \varepsilon^{k-4-2\kappa/d}\right).
$$

With the L^2-result of Theorem 4.15, we find for any $\ell > 1$ a constant $C_\ell > 0$ such that

$$\mathbb{P}\left(\|v(\tau_\varepsilon)\| > \varepsilon^{k-2}\right) \leq C_\ell \varepsilon^\ell .$$

Moreover, by Lemma 4.19 we derive with Chebychev's inequality

$$\begin{aligned}
\mathbb{P}\left(\|\nabla v(\tau_\varepsilon)\|_{L^2} > \varepsilon^{k-4-2\kappa/d}\right) &\leq C\varepsilon^{-2p(k-4+2\kappa/d)}\,\mathbb{E}\|\nabla v(\tau_\varepsilon)\|_{L^2}^{2p} \\
&\leq C\varepsilon^{-2p(k-4-2\kappa/d)}\varepsilon^2\left[\varepsilon^{-2}+T_\varepsilon\right]a_\varepsilon^p \\
&= C\left(\varepsilon^{-2k+8+4\kappa/d}a_\varepsilon\right)^p\left[1+\varepsilon^2 T_\varepsilon\right] \\
&= C\left(\varepsilon^{-2k+6+4\kappa/d}\,\Gamma_\varepsilon\right)^p\left[1+\varepsilon^2 T_\varepsilon\right].
\end{aligned}$$

With the definition (4.8) of Γ_ε and the assumptions on the noise strength, we see that

$$\varepsilon^{-2k+6+4\kappa/d}\,\Gamma_\varepsilon \leq C\varepsilon^{-2k+6+4\kappa/d}(\varepsilon^{2k-6}+\varepsilon^{-3}\eta_0^2+\varepsilon^{-2}\eta_0+\eta_2) \leq \varepsilon^{4\kappa/d}.$$

Now, choosing p large enough yields the result. □

Due to the scaling of the radii in Theorem 4.20, we can finally rephrase the main stability result in terms of the H^1_ε-norm, which is given by (4.4).

Corollary 4.21. *Under the assumptions of Theorem 4.20 and for any sufficiently large $C > \nu$ and any $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that*

$$\mathbb{P}\left(\|v(t)\|_{H^1_\varepsilon} < C\varepsilon^{k-3-\kappa} \;\; \forall t \in [0, \varepsilon^{-N}]\right) \geq 1 - C_N \varepsilon^N .$$

CHAPTER 5

Multiple kinks for the Allen–Cahn equation in one space dimension

In this chapter, we study the stochastic Allen–Cahn equation (AC) together with its mass conserving modification (mAC) posed on an one-dimensional domain and introduce an additive spatially smooth and white in time noise $\partial_t W$

$$\begin{cases} \partial_t u \,=\, \varepsilon^2 u_{xx} - f(u) + \partial_t W, & 0 < x < 1,\ t > 0 \\ u_x \,=\, 0, & x \in \{0,1\}. \end{cases} \tag{AC}$$

Here, $0 < \varepsilon \ll 1$ is a small parameter measuring the typical width of a phase transition, and $f = F'$ is the derivative of a double well potential F. We assume that $F \in C^3(\mathbb{R})$ is a smooth, even potential satisfying

(S1) $F(u) \geq 0$ and $F(u) = 0$ if, and only if, $u = \pm 1$,

(S2) F' has three zeros $\{0, \pm 1\}$ and $F''(0) < 0, F''(\pm 1) > 0$,

(S3) F is symmetric: $F(u) = F(-u)\ \ \forall u \geq 0$.

The standard example is $F(u) = \frac{1}{4}(1-u^2)^2$ and thus $f(u) = u^3 - u$. As before for the simplicity of some arguments, we focus for the most part of this chapter on this standard quartic potential, although the results remain valid for potentials satisfying the conditions (S1)–(S3). In Section 5.6, we treat a more general class of nonlinearities given by polynomials of odd degree with positive leading order term.

For the moment, let us assume that $\int_0^1 \dot{W}(t,x)\,\mathrm{d}x = 0$ for all $t \geq 0$, i.e., in a Fourier series expansion there is no noise on the constant mode. In contrast to the Cahn–Hilliard equation, (AC) does not preserve mass as

$$\partial_t \int_0^1 u(t,x)\,\mathrm{d}x \,=\, \varepsilon^2 \int_0^1 u_{xx}\,\mathrm{d}x - \int_0^1 f(u)\,\mathrm{d}x + \int_0^1 \dot{W}(t,x)\,\mathrm{d}x \,=\, -\int_0^1 f(u)\,\mathrm{d}x.$$

Throughout our analysis, we will therefore separately consider the mass conserving Allen–Cahn equation (mAC)

$$\begin{cases} \partial_t u \,=\, \varepsilon^2 u_{xx} - f(u) + \displaystyle\int_0^1 f(u)\,\mathrm{d}x + \partial_t W, & 0 < x < 1,\ t > 0 \\ u_x \,=\, 0, & x \in \{0,1\}, \end{cases} \tag{mAC}$$

where we added the integral of f over the interval $(0,1)$ to guarantee the conservation of mass.

For the remainder of this chapter, we introduce the following notation. We denote the standard inner product in $L^2(0,1)$ by $\langle \cdot, \cdot \rangle$, i.e., $\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$, and the L^2-norm by $\|\cdot\|$. Other scalar products and norms appearing in subsequent sections will be endowed with a subindex. Moreover, we denote the Allen–Cahn operator by

$$\mathcal{L}(\psi) \,=\, \varepsilon^2 \psi_{xx} - f(\psi).$$

As in our previous applications in higher space dimensions, we consider for a given ansatz function u^h (defined later in Definition 5.5) the Taylor expansion of $\mathcal{L}$ around u^h

$$\mathcal{L}(u^h + \psi) = \mathcal{L}(u^h) + \mathcal{L}^h\psi + \mathcal{N}^h(\psi),$$

where we define the linearization $\mathcal{L}^h$ of $\mathcal{L}$ at the ansatz function u^h and the remaining nonlinear terms $\mathcal{N}^h(v)$ by

$$\mathcal{L}^h\psi := \mathrm{D}\mathcal{L}(u^h)\psi = \varepsilon^2\psi_{xx} - f'(u^h)\psi \quad \text{and} \quad \mathcal{N}^h(\psi) := f(u^h) - f(u^h+\psi) + f'(u^h)\psi.$$

In the prototypical case of the quartic potential, this leads to $\mathcal{L}^h\psi = \varepsilon^2\psi_{xx} + \psi - 3(u^h)^2\psi$ and $\mathcal{N}^h(\psi) = -3u^h\psi^2 - \psi^3$.

In the case of the mass conserving Allen–Cahn equation, we have to assume that the Wiener process W has mean zero. Furthermore, we need that solutions to both (AC) and (mAC) are sufficiently smooth in space and hence, we need that the stochastic forcing $\partial_t W$ is sufficiently smooth in space, too. In the remainder of this chapter, we will assume that W is given by a $\mathcal{Q}$-Wiener process satisfying the following regularity properties.

Assumption 5.1 (Regularity of the Wiener process W)**.**
Let W be a $\mathcal{Q}$-Wiener process in the underlying Hilbert space $L^2(\Omega)$, $\mathcal{Q}$ a symmetric operator and $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis with corresponding eigenvalues α_k^2 such that

$$\mathcal{Q}e_k = \alpha_k^2 e_k \qquad \text{and} \qquad W(t) = \sum_{k\in\mathbb{N}} \alpha_k\beta_k(t)e_k,$$

for a sequence of independent real-valued standard Brownian motions $\{\beta_k(t)\}_{k\in\mathbb{N}}$.
We assume that the $\mathcal{Q}$-Wiener process W satisfies

$$\operatorname{trace}_{L^2}(\mathcal{Q}) = \sum_{k\in\mathbb{N}} \alpha_k^2 =: \eta_0 < \infty.$$

Moreover, in the case of the mass conserving Allen–Cahn equation (mAC), we suppose that W takes its values in $L_0^2(\Omega)$, that is,

$$\int_0^1 W(t,x)\,\mathrm{d}x = 0 \quad \textit{for all } t \geq 0.$$

Note that our results will thus depend on the squared noise strength η_0, which also depends on the parameter $\varepsilon > 0$. The exact size of η_0 will be fixed in subsequent sections.

5.1 Construction of the slow manifold

In this section, we construct the fundamental building block for our analysis, the slow manifolds $\mathcal{M}$ and $\mathcal{M}_\mu$. Our construction of the slow manifolds is different to [CP89], which is the deterministic case. Opposed to this work, we do not introduce a cut-off function to glue together the profiles connecting the stable phases ±1. With this cut-off function, the authors took extra care of the exponentially small error away from the interface positions, which is crucial as the motion of the kinks is dominated by exponentially small terms. In our stochastic case, however, the (polynomial in ε) noise strength dominates and hence, we are not concerned with these exponentially small terms. The main idea in our construction goes as follows:

We start with a stationary solution U to (AC) on the whole line $\mathbb{R}$, centered at 0 and connecting the stable phases -1 and $+1$ (Definition 5.2). Using the exponential decay of U (Proposition 5.3), we introduce a rescaled version in the domain $[0,1]$ in order to construct an ansatz function u^h, which jumps from ±1 to ∓1 in an $\mathcal{O}(\varepsilon)$-neighborhood of the zeros h_i (Definition 5.5).

Throughout our analysis, we fix the number $N+1$ of transitions. The presented results hold up to times, where the distance between two neighboring interfaces gets too small and we thus cannot exclude the possibility of a collapse of two interfaces. This behavior of the stochastic equation was not studied in full detail yet. In the deterministic case, we refer to the nice work by X. Chen, [Che04]. For some ideas in the stochastic case, see the doctoral thesis of Weber [Web14]. Essentially, after an annihilation the number of transitions is reduced to $N-1$ and we can restart our analysis on a lower-dimensional slow manifold.

Definition 5.2 (The heteroclinic)**.**
Let U be the unique, increasing solution to

$$U'' - f(U) \,=\, 0, \quad U(0) \,=\, 0, \quad \lim_{x\to\pm\infty} U(x) \,=\, \pm 1. \tag{5.1}$$

In the prototypical case $f(u) = u^3 - u$, we have the explicit solution $U(x) = \tanh(\frac{x}{\sqrt{2}})$.

The function U is the heteroclinic of the ODE connecting the stable points -1 and $+1$. For a later discussion of the spectrum of the linearized Allen–Cahn operator, we need some relations between the heteroclinic U and the potential F. We observe that if U is a solution to (5.1), then

$$\partial_x \left(\frac{U_x^2}{2} - F(U) \right) \,=\, U_x \left(U_{xx} - F'(U) \right) \,=\, 0.$$

From the boundary condition $U(0) = 0$, we conclude that solving equation (5.1) is equivalent to solving the first-order ODE

$$U_x \,=\, \sqrt{2F(U)}, \quad U(0) \,=\, 0, \quad \lim_{x\to\pm\infty} U(x) \,=\, \pm 1. \tag{5.2}$$

By the assumptions on the potential F, we see that $\sqrt{F}$ is C^1 and hence, the solution to (5.2) is unique. Moreover, we observe that all derivatives of U can be expressed as a function of U. For instance, we have $U'' = F'(U)$, $U^{(3)} = F''(U)\sqrt{2F(U)}$, and so on. Also note that, due to the symmetry of F, the mirrored function $-U$ solves the same differential equation, but transits from $U(-\infty) = +1$ to $U(+\infty) = -1$. For some fine properties of U, we refer to the work of Carr and Pego [CP89], which is based on [CGS84]. Crucial for the construction of a slow manifold (cf. Definition 5.8) is that the heteroclinic U together with its derivatives decay exponentially fast. The following proposition can be shown via phase plane analysis. For a proof we refer to [AFS96].

Proposition 5.3 (Exponential decay of U)**.**
Let $U(x)$, $x \in \mathbb{R}$, be the heteroclinic defined by (5.1). There exist constants $c, C > 0$ such that for $x \geq 0$

$$|1 \mp U(\pm x)| \,\leq\, Ce^{-cx}, \quad |U'(\pm x)| \,\leq\, Cce^{-cx}, \quad \textit{and} \quad |U''(\pm x)| \,\leq\, Cc^2 e^{-cx}.$$

For $\xi \in \mathbb{R}$, we define a translated and rescaled version of U by

$$U(x;\, \xi,\, \pm 1) \,\coloneqq\, \pm\, U \left(\frac{x - \xi}{\varepsilon} \right). \tag{5.3}$$

One easily verifies that $U(\cdot\,; \xi, \pm 1)$ is a solution to the rescaled ODE $\varepsilon^2 U_{xx} - f(U) = 0$, centered at $U(\xi; \xi, \pm 1) = 0$ and going from ∓ 1 to ± 1. Due to the exponential decay of the heteroclinic, the rescaled profile $U(x\,; \xi, \pm 1)$ is exponentially close to the states ± 1, if x is at least $\mathcal{O}(\varepsilon^{1-})$-away from the zero ξ.

Lemma 5.4. *Let $\kappa > 0$ and $0 < \varepsilon < \varepsilon_0$. Then, uniformly for $x > \xi + \varepsilon^{1-\kappa}$,*

$$U(x;\, \xi,\, \pm 1) \;=\; \pm 1 + \mathcal{O}(\exp)$$

and, uniformly for $x < \xi - \varepsilon^{1-\kappa}$,

$$U(x;\, \xi,\, \pm 1) \;=\; \mp 1 + \mathcal{O}(\exp).$$

Similar exponential estimates hold for the derivatives of $U(\cdot\,;\, \xi, \pm 1)$.

Proof. By Proposition 5.3 and monotonicity of U, we obtain for $x > \xi + \varepsilon^{1-\kappa}$:

$$U(x;\, \xi,\, +1) \;>\; U(\varepsilon^{-\kappa}) \;\geq\; 1 - C\exp(-c\varepsilon^{-\kappa}).$$

The other cases work in the same way. □

Motivated by this lemma, we can construct for $h \in (0,1)^{N+1}$ with $h_1 < h_2 < \ldots < h_{N+1}$ profiles $u^h : [0,1] \to \mathbb{R}$ such that u^h jumps from ± 1 to ∓ 1 in a small neighborhood around h_i of size $\mathcal{O}(\varepsilon)$. Locally around h_i, we assume that

$$u^h(x) \;\approx\; U(x\,;\, h_i, (-1)^{i+1}).$$

If the distance between two neighboring interfaces and the distance to the boundary is bounded from below by $\varepsilon^{1-\kappa}$ for some small $\kappa > 0$, we assured in Lemma 5.4 that each profile $U(x;\, h_i,\, \pm 1)$ reaches ± 1 up to an exponentially small error. Thus, we can essentially define u^h as the sum of profiles given by (5.3). This leads to the following definition.

Definition 5.5 (The profile u^h)**.**
Fixing $\rho = \varepsilon^{\kappa}$ for $\kappa > 0$ very small, we define the set Ω_ρ of admissible interface positions in the interval $(0,1)$ by

$$\Omega_\rho \;\coloneqq\; \left\{ h \in \mathbb{R}^{N+1} \;:\; 0 \;<\; h_1 \;<\; \ldots \;<\; h_{N+1} \;<\; 1,\; \max_{j=0,\ldots,N+1} |h_{j+1} - h_j| \;>\; \varepsilon/\rho \right\},$$

where $h_0 \coloneqq -h_1$ and $h_{N+2} \coloneqq 2 - h_{N+1}$. For $x \in (0,1)$ and $h \in \Omega_\rho$, we define

$$u^h(x) \;\coloneqq\; \sum_{j=1}^{N+1} U\left(x;\, h_j,\, (-1)^{j+1}\right) + \beta_N(x),$$

where the normalization function $\beta_N(x)$ satisfies $\beta_N(x) = \frac{(-1)^N - 1}{2} + \mathcal{O}(\exp)$ (cf. Remark 5.6).

Note that the positions h_0 and h_{N+2} were introduced to bound the distance of the interface positions from the boundary 0 and 1. Moreover, it is straightforward to check that the set Ω_ρ is convex.

Remark 5.6. Let us comment on why we needed to add the normalization term $\beta_N(x)$ in the definition of u^h. Due to symmetry, we can assume that the multi-kink profile starts in the phase $u^h(0) = -1$. Depending on the parity of the number of transitions, we have to add a constant to assure this. As $h_j > \varepsilon/\rho$, we obtain by Lemma 5.4 in $x = 0$

$$\sum_{j=1}^{N+1} U\left(0;\, h_j,\, (-1)^{j+1}\right) \;=\; \sum_{j=1}^{N+1} (-1)^j + \mathcal{O}(\exp) \;=\; -1 + \frac{1 - (-1)^N}{2} + \mathcal{O}(\exp).$$

Therefore, we have to add the correction $\frac{1}{2}((-1)^N - 1) + \mathcal{O}(\exp)$ to obtain $u^h(0) = -1$. Moreover, we need to assure that u^h satisfies Neumann boundary conditions.

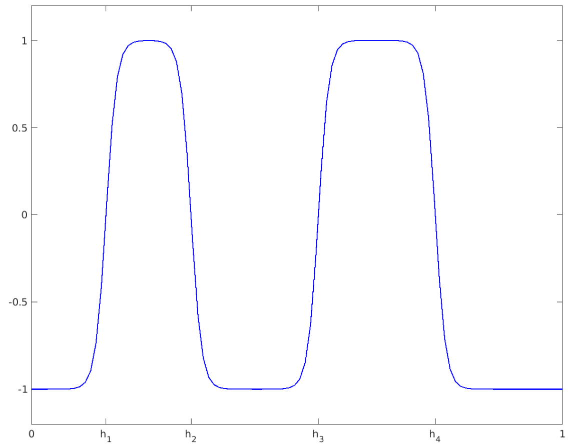

Figure 5.1.: The profile u^h

By Lemma 5.4, the derivative of $U(x;\, h_j, \pm 1)$ is exponentially small at $x \in \{0,1\}$. Hence, in order to correct the boundary condition, we additionally have to add a function of order $\mathcal{O}(\exp)$.

Before we finally define the slow manifolds for the (mass conserving) Allen–Cahn equation (Definition 5.8), we collect some properties of the multi-kink configurations u^h.

Proposition 5.7 (Properties of u^h)**.**
The function u^h is an almost stationary solution to (AC) in the sense that it fails the equation by an exponentially small error, that is,

$$\varepsilon^2 u^h_{xx} - f(u^h) \;=\; \mathcal{O}(\exp), \qquad u^h_x(0)\,,\, u^h_x(1) \;=\; 0. \tag{5.4}$$

For $i, j \in \{1, \dots, N+1\}$, we denote the partial derivatives of u^h with respect to the h–variables by $u^h_i = \partial_{h_i} u^h$, $u^h_{ij} = \partial_{h_i}\partial_{h_j} u^h$, and third derivatives accordingly. We have

$$u^h_i(x) \;=\; U'(x;\, h_i,\, (-1)^{i+1}) + \mathcal{O}(\exp) \;=\; (-1)^i \varepsilon^{-1} U'\left(\frac{x-h_i}{\varepsilon}\right) + \mathcal{O}(\exp). \tag{5.5}$$

Furthermore, the following estimates hold true in $L^2(0,1)$:

$$\begin{aligned} \langle u^h_i, u^h_j \rangle \;&=\; \mathcal{X}\varepsilon^{-1}\delta_{ij} + \mathcal{O}(\exp), & \|u^h_{ij}\| \;&=\; \mathcal{O}(\varepsilon^{-3/2})\delta_{ij} + \mathcal{O}(\exp), \\ \langle u^h_{kk}, u^h_k \rangle \;&=\; \mathcal{O}(\exp), \qquad \textit{and} & \|u^h_{kkk}\| \;&=\; \mathcal{O}(\varepsilon^{-5/2}), \end{aligned}$$

where $\mathcal{X} := \int_{\mathbb{R}} U'(y)^2\,\mathrm{d}y$. In $L^\infty(0,1)$, we have

$$\|u^h\|_\infty \;=\; \mathcal{O}(1) \qquad \textit{and} \qquad \|u^h_i\|_\infty \;=\; \mathcal{O}(\varepsilon^{-1}).$$

Proof. By Lemma 5.4 we derive for $|x - h_i| < \varepsilon/\rho$

$$\begin{aligned} u^h(x) \;&=\; U(x\,;\, h_i, (-1)^{i+1}) + \sum_{j \neq i} U(x\,;\, h_i, (-1)^{j+1}) + \tfrac{(-1)^N - 1}{2} + \mathcal{O}(\exp) \\ &=\; U(x\,;\, h_i, (-1)^{i+1}) + \sum_{j<i} (-1)^{j+1} + \sum_{j>i} (-1)^j + \tfrac{(-1)^N - 1}{2} + \mathcal{O}(\exp) \\ &=\; U(x\,;\, h_i, (-1)^{i+1}) + 1 + \tfrac{(-1)^i - 1}{2} + \tfrac{1 - (-1)^N}{2} + \tfrac{(-1)^{i+1} - 1}{2} + \tfrac{(-1)^N - 1}{2} + \mathcal{O}(\exp) \\ &=\; U(x\,;\, h_i, (-1)^{i+1}) + \mathcal{O}(\exp). \end{aligned}$$

As $U(\cdot\,; h_i, (-1)^{i+1})$ solves $\varepsilon^2 U'' - f(U) = 0$, we verified (5.4) locally around each h_i. Lemma 5.4 implies that $\mathcal{O}(\varepsilon^{1-})$-away from the interface

$$u^h_{xx} = \mathcal{O}(\exp) \quad \text{and} \quad f(u^h) = f(\pm 1 + \mathcal{O}(\exp)) = \mathcal{O}(\exp).$$

Equation (5.5) follows directly from Definition 5.5, (5.3) and Lemma 5.4. By Lemma 5.4, we also see that $U'(x\,; h_i, (-1)^{i+1})$ is exponentially small for $|x - h_i| > \varepsilon/\rho$ and thus we obtain $\langle u^h_i, u^h_j \rangle = \mathcal{O}(\exp)$ for $i \neq j$. Moreover, the same argument implies that higher derivatives with respect to different positions h_i and h_j are exponentially small. The L^∞-bounds follow directly from the definition of u^h and (5.5). The L^2-norm of u^h_k is given by

$$\begin{aligned}
\|u^h_k\|^2 &= \varepsilon^{-2} \int_0^1 U'\left(\frac{x-h_k}{\varepsilon}\right)^2 \mathrm{d}x + \mathcal{O}(\exp) \\
&= \varepsilon^{-1} \int_{-h_k/\varepsilon}^{(1-h_k)/\varepsilon} U'(y)^2 \,\mathrm{d}y + \mathcal{O}(\exp) \;=\; \varepsilon^{-1} \int_{\mathbb{R}} U'(y)^2 \,\mathrm{d}y + \mathcal{O}(\exp).
\end{aligned}$$

In the last step, we used that $h \in \Omega_\rho$ and $|U(x)| \leq c e^{-c|x|}$ by Proposition 5.3. Thus, we obtain

$$\int_{-\infty}^{-h_k/\varepsilon} U'(y)^2 \,\mathrm{d}y \;\leq\; \int_{-\infty}^{-1/\rho} U'(y)^2 \,\mathrm{d}y \;\leq\; c \int_{-\infty}^{-1/\rho} e^{-c|y|} \,\mathrm{d}y \;=\; \mathcal{O}(\exp),$$

and with the same argument

$$\int_{(1-h_k)/\varepsilon}^{\infty} U'(y)^2 \,\mathrm{d}y \;=\; \mathcal{O}(\exp).$$

Analogously, the n-th derivative with respect to h_k is then estimated by

$$\begin{aligned}
\|\partial^n_{h_k} u^h\|^2 &= \varepsilon^{-2n} \int_0^1 U^{(n)}\left(\frac{x-h_k}{\varepsilon}\right)^2 \mathrm{d}x + \mathcal{O}(\exp) \\
&= \varepsilon^{-2n+1} \int_{\mathbb{R}} U^{(n)}(y)^2 \,\mathrm{d}y + \mathcal{O}(\exp) \;=\; \mathcal{O}(\varepsilon^{-2n+1}).
\end{aligned}$$

The mixed term can be estimated as follows:

$$\begin{aligned}
\langle u^h_{kk}, u^h_k \rangle &= \varepsilon^{-3} \int_0^1 U''\left(\frac{x-h_k}{\varepsilon}\right) U'\left(\frac{x-h_k}{\varepsilon}\right) \mathrm{d}x + \mathcal{O}(\exp) \\
&= \varepsilon^{-2} \int_{-h_k/\varepsilon}^{(1-h_k)/\varepsilon} U''(x) U'(x) \,\mathrm{d}x + \mathcal{O}(\exp) \\
&= \frac{1}{2}\varepsilon^{-2} \left[U'\left(\frac{1-h_k}{\varepsilon}\right)^2 - U'\left(-\frac{h_k}{\varepsilon}\right)^2 \right] + \mathcal{O}(\exp) \;=\; \mathcal{O}(\exp).
\end{aligned}$$

□

We finally introduce approximate slow manifolds for the stochastic (mass conserving) Allen–Cahn equation. The second manifold will play an important role in the study of the mass conserving Allen–Cahn equation (mAC), while the first one will be used for the analysis of (AC) without this constraint.

Definition 5.8 (Slow manifolds).
For Ω_ρ and u^h given by Definition 5.5, we define the *approximate slow manifold* by

$$\mathcal{M} := \left\{ u^h \,:\, h \in \Omega_\rho \right\}.$$

Fixing a mass $\mu \in (-1, 1)$, we define the *mass conserving approximate manifold* by

$$\mathcal{M}_\mu := \left\{ u^h \in \mathcal{M} \,:\, \int_0^1 u^h(x) \,\mathrm{d}x \;=\; \mu \right\}.$$

Note that we have a global chart for $\mathcal{M}$. Later, we will see that this also holds true for $\mathcal{M}_\mu$, as it is the manifold $\mathcal{M}$ intersected by a vector space of codimension 1 (see Lemma 5.9).

We have to compute the tangent vectors for $\mathcal{M}$ and $\mathcal{M}_\mu$, since we need them later in Definition 5.17 to define a coordinate system around the slow manifolds (see also Definition 2.2 in the general framework). We immediately see that the tangent space of the slow manifold $\mathcal{M}$ at u^h with $h \in \Omega_\rho$ is given by

$$\mathcal{T}_{u^h}\mathcal{M} \,=\, \operatorname{span}\left\{u_i^h \,:\, i \,=\, 1\,,\,\ldots\,,\,N+1\right\}.$$

With Proposition 5.7 one checks readily that the tangent vectors u_i^h have essentially (up to an exponentially small error) disjoint support and therefore, $\mathcal{T}_{u^h}\mathcal{M}$ is non-degenerate and has full dimension $N+1$. For the second manifold $\mathcal{M}_\mu$ we will see in the following lemma that, due to mass conservation, it is possible to reduce the parameter space Ω_ρ by one dimension.

Lemma 5.9. *There is a smooth map $h_{N+1} : \mathbb{R}^N \to \mathbb{R}$ such that*

$$u^h \in \mathcal{M}_\mu \quad\Longleftrightarrow\quad h = (\xi, h_{N+1}(\xi)) \in \Omega_\rho \quad \textit{with} \quad \xi \,\coloneqq\, (h_1\,,\,\ldots\,,h_N).$$

Moreover, the partial derivatives of h_{N+1} with respect to h_i, $i = 1, \ldots, N$, are given by

$$\frac{\partial h_{N+1}}{\partial h_i} \,=\, (-1)^{N-i} + \mathcal{O}(\exp).$$

Proof. In the mass conserving case, the set of admissible positions in Ω_ρ is the zero set of the smooth map $\Omega_\rho \ni h \mapsto \int_0^1 u^h(x)\,\mathrm{d}x - \mu$. Proposition 5.7 implies that

$$\begin{aligned}\partial_{h_{N+1}} \int_0^1 u^h(x)\,\mathrm{d}x \,&=\, \int_0^1 u^h_{N+1}(x)\,\mathrm{d}x \,=\, \mathcal{O}(\exp) + (-1)^{N+1}\varepsilon^{-1}\int_0^1 U'\left(\frac{x-h_{N+1}}{\varepsilon}\right)\mathrm{d}x \\ &=\, \mathcal{O}(\exp) + (-1)^{N+1}\left[U\left(1\,;\,h_{N+1},(-1)^N\right) - U\left(0\,;\,h_{N+1},(-1)^N\right)\right] \\ &=\, 2(-1)^{N+1} + \mathcal{O}(\exp).\end{aligned}$$

In the last step, we used that h lies in Ω_ρ and thus by Lemma 5.4

$$U\left(1\,;\,h_{N+1},(-1)^N\right) \,=\, 1+\mathcal{O}(\exp) \quad \text{and} \quad U\left(0\,;\,h_{N+1},(-1)^N\right) \,=\, -1+\mathcal{O}(\exp).$$

By the implicit function theorem, we can then write h_{N+1} as a smooth function of the first N interface positions $(h_1, \ldots, h_N)$. We compute

$$0 \,=\, \partial_{h_i}\mu \,=\, \int_0^1 u_i^h\,\mathrm{d}x + \frac{\partial h_{N+1}}{\partial h_i}\int_0^1 u^h_{N+1}\,\mathrm{d}x \,=\, 2(-1)^i + \mathcal{O}(\exp) + 2(-1)^{N+1}\frac{\partial h_{N+1}}{\partial h_i},$$

and therefore, the partial derivative of h_{N+1} with respect to h_i is given by

$$\frac{\partial h_{N+1}}{\partial h_i} \,=\, (-1)^{N-i} + \mathcal{O}(\exp).$$

□

Motivated by Lemma 5.9, we define

$$\xi \,\coloneqq\, (\xi_1, \ldots, \xi_N) \,=\, (h_1, \ldots, h_N)$$

and consider $h_{N+1} = h_{N+1}(\xi)$ as a function of ξ. We can then write

$$\mathcal{M}_\mu \,=\, \left\{u^h \,:\, h \in \mathcal{A}_\rho\right\} \quad \text{with} \quad \mathcal{A}_\rho \,\coloneqq\, \left\{(\xi, h_{N+1}(\xi)) \in \Omega_\rho \,:\, \xi \in [0,1]^N\right\}. \tag{5.6}$$

In the sequel, we denote the elements of $\mathcal{M}_\mu$ by u^ξ. As before, we denote the partial derivatives of u^ξ with respect to ξ_i by u_i^ξ, and higher derivatives accordingly.

The tangent space of the mass conserving manifold $\mathcal{M}_\mu$ at u^ξ is given by

$$\mathcal{T}_{u^\xi}\mathcal{M}_\mu = \operatorname{span}\left\{u_i^\xi = u_i^h + (-1)^{N-i}u_{N+1}^h + \mathcal{O}(\exp) : i = 1,\ldots,N\right\}.$$

Here, we used that by the chain rule and Lemma 5.9

$$\frac{\partial u^\xi}{\partial \xi_i} = \frac{\partial u^h}{\partial h_i} + \frac{\partial h_{N+1}}{\partial h_i}\cdot\frac{\partial u^h}{\partial h_{N+1}} = \frac{\partial u^h}{\partial h_i} + (-1)^{N-i}\frac{\partial u^h}{\partial h_{N+1}} + \mathcal{O}(\exp).$$

This is a linear combination of tangent vectors of $\mathcal{M}$. Since the functions u_i^h span an $(N+1)$-dimensional space and the transformation matrix converting these functions into $\{u_1^\xi,\ldots,u_N^\xi\}$ has full rank N, we immediately obtain that the tangent space $\mathcal{T}_{u^h}\mathcal{M}_\mu$ is non-degenerate as well.

5.2 The linearized Allen–Cahn operator

Important for the stability of the slow manifolds are spectral estimates concerning the linearization of the Allen–Cahn operator at a multi-kink configuration. In more detail, for v orthogonal to the tangent space of $\mathcal{M}_{(\mu)}$, we aim to bound the quadratic form $\langle \mathcal{L}^h v, v\rangle$ (compare to Assumption 2.1). First, we consider the singular Sturm–Liouville problem

$$Ly = y'' - f'(U)y = -\lambda y \tag{5.7}$$

in $L^2(\mathbb{R})$, where U is the heteroclinic solution defined by (5.1). Note that the ODE (5.1) directly implies that U' is an eigenfunction of L corresponding to the eigenvalue zero. As $U' > 0$, we also know that zero must be the largest eigenvalue. The following description of the spectral behavior of L orthogonal to U' is taken from [OR07], Proposition 3.2.

Lemma 5.10 (Spectral gap of the Allen–Cahn operator, [OR07], Proposition 3.2)**.**
There exists a constant $\lambda_0 > 0$ such that if $v \in H^1(\mathbb{R})$ satisfies

$$\text{(i) } v(0) = 0 \qquad \textit{or} \qquad \text{(ii) } \int_{\mathbb{R}} v(s)U'(s)\,\mathrm{ds} = 0,$$

then it holds true that

$$\langle Lv, v\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left[-v'(s)^2 - f'(U(s))v(s)^2\right]\,\mathrm{ds} \le -\lambda_0\|v\|^2_{L^2(\mathbb{R})}.$$

In case of the toy problem $f(u) = u^3 - u$, we have $\lambda_0 = 3/2$ (cf. [AFS96]). After some lengthy calculation, one can show that in this case $U\sqrt{U'}$ serves as an eigenfunction of (5.7) corresponding to the eigenvalue $-\lambda_0$. The eigenfunction $U\sqrt{U'}$ has exactly one zero and hence it corresponds to the second largest eigenvalue. As $\lim_{|x|\to\infty}(1-3U^2) = -2$, we also know by a standard argument for Schrödinger operators that the essential spectrum lies in the interval $(-\infty,-2]$. For more details on the spectrum of Schrödinger operators, we refer to [HS96]. The standard arguments for Sturm–Liouville problems can be found in [Wal98].

For the study of the spectral gap of the linearized Allen–Cahn operator in Theorem 5.14, we need to ensure that the constant λ_0 is bounded by the supremum of the function f' over the interval $(-1,1)$.

Assumption 5.11.
Let λ_0 be the constant from Lemma 5.10. We assume that $\lambda_0 < \sup_{x\in(-1,1)} f'(x)$.

As mentioned, the function $v = U\sqrt{U'}$ serves as an eigenfunction to the quartic potential F corresponding to $\lambda_0 = 3/2 < 2 = \sup_{x\in(-1,1)} f'(x)$. By evaluating the quadratic form $\langle Lv, v\rangle$ in the general case, one gets a sufficient condition for Assumption 5.11.

Lemma 5.12. *Let F be a smooth even double well satisfying the conditions (S1)-(S3). Moreover, suppose that*

$$\int_{-1}^{1} \frac{3}{2} s^2 F''(s) + \frac{1}{8} \frac{s^2 F'(s)^2}{F(s)} \,\mathrm{d}s \;<\; \frac{2}{3} \sup_{x \in (-1,1)} F''(x).$$

Then, Assumption 5.11 holds true.

Proof. Let $v := U\sqrt{U}$, where U is defined by (5.1). With the relations $U''(x) = F'(U(x))$ and $U'(x) = \sqrt{2F(U(x))}$ (see the comment after Definition 5.3), one computes that

$$\begin{aligned} \int_{\mathbb{R}} v_x^2 + F''(U) v^2 \,\mathrm{d}x \;&=\; \int_{\mathbb{R}} -UF''(U)U' + F''(U)U^2U' + \frac{U^2 F'(U)^2}{8F(U)} U' \,\mathrm{d}x \\ = \int_{-1}^{1} -sF''(s) + F''(s)s^2 + \frac{s^2 F'(s)^2}{8F(s)} \,\mathrm{d}s \;&=\; \int_{-1}^{1} \frac{3}{2} s^2 F''(s) + \frac{1}{8} \frac{s^2 F'(s)^2}{F(s)} \,\mathrm{d}s. \end{aligned} \tag{5.8}$$

In the last step, we utilized that the potential F, and thus F'', is a symmetric function. Therefore, the integral of the antisymmetric function $sF''(s)$ vanishes. Since $v(0) = 0$, we have by Lemma 5.10

$$\int_{\mathbb{R}} v_x^2 + F''(U)v^2 \,\mathrm{d}x \;\geq\; \lambda_0 \|v\|^2 = \lambda_0 \int_{\mathbb{R}} U^2 U' \,\mathrm{d}x \;=\; \lambda_0 \int_{-1}^{1} x^2 \,\mathrm{d}x \;=\; \frac{2}{3}\lambda_0. \tag{5.9}$$

Combining (5.8) and (5.9) immediately yields the assertion of the lemma. □

Instead of testing with the function $U\sqrt{U'}$, we can evaluate the eigenvalue problem (5.7) in terms of general functions having exactly one zero at $x = 0$. If existent, we can compute the second eigenvalue explicitly. This will provide us with a necessary condition for Assumption 5.11.

Lemma 5.13. *Assume that the eigenvalue problem (5.7) possesses a second eigenvalue $\lambda_0 > 0$. Then, λ_0 is given by*

$$\lambda_0 \;=\; \frac{1}{2} f'(\pm 1) + \frac{1}{8} \lim_{x \to \pm 1} \frac{f(x)^2}{F(x)}.$$

In particular, Assumption 5.11 can only hold true if

$$\lim_{x \to \pm 1} \frac{f(x)^2}{F(x)} \;<\; 4f'(\pm 1).$$

Proof. Let $v(x)$ be the eigenfunction associated to the eigenvalue $-\lambda_0$. Since we can always assume that v attains its only zero at $x = 0$, we can represent v as $v = g(U)\sqrt{U'}$ for some smooth, nonnegative function g satisfying $g(0) = 0$. Plugging v into (5.7), we obtain

$$\begin{aligned} \lambda_0 v \;=\; Lv \;=\; &g''(U)(U')^{5/2} + 2g'(U)(U')^{1/2}U'' - \frac{1}{4} g(U)(U')^{-3/2}(U'')^2 \\ &+ \frac{1}{2} g(U)(U')^{-1/2}U''' - f'(U)g(U)(U')^{1/2}. \end{aligned}$$

Multiplying this equation by $(U')^{-1/2}$ yields

$$\begin{aligned} \lambda_0 g(U) \;&=\; g''(U)(U')^2 + 2g'(U)U'' - \frac{1}{4} g(U) \left(\frac{U''}{U'}\right)^2 + \frac{1}{2} g(U) \frac{U'''}{U'} - g(U)f'(U) \\ &=\; 2g''(U)F(U) + 2g'(U)f(U) - \frac{1}{8} g(U) \frac{f(U)^2}{F(U)} - \frac{1}{2} g(U)f'(U). \end{aligned} \tag{5.10}$$

In the last step, we utilized the relations $U' = \sqrt{2F(U)}$, $U'' = f(U)$, and $U''' = f'(U)U'$. By assumptions (S1) and (S2) on the potential F, we observe that $F(\pm 1) = f(\pm 1) = 0$. Hence, in the limit $x \to \pm\infty$, we derive from (5.10)

$$\lambda_0\, g(\pm 1) \;=\; -\frac{1}{8} g(\pm 1) \lim_{x\to\pm 1} \frac{f(x)^2}{F(x)} - \frac{1}{2} g(\pm 1) f'(\pm 1).$$

Dividing by $g(\pm 1) \neq 0$ concludes the proof. □

With the spectral gap of Lemma 5.10 at hand, we consider the linearization of the Allen–Cahn operator at a multi-kink state $u^h \in \mathcal{M}$. The following theorem gives a bound on the quadratic form orthogonal to the tangent space $\mathcal{T}_h\mathcal{M}$. Essentially, up to exponentially small terms, the support of the tangent vectors u_i^h is concentrated in a small neighborhood of width ε around the zero h_i. Hence, it is sufficient to study the quadratic form locally around each h_i. After rescaling, we essentially arrive at the setting of Theorem 5.10 and the spectral gap of order $\mathcal{O}(1)$ is transferred to our problem.

Theorem 5.14 (Spectral gap for (AC))**.**
Let $u^h \in \mathcal{M}$ and suppose that Assumption 5.11 holds true. Moreover, assume that $v \perp u_i^h$ for any $i = 1, \dots, N+1$. Then, for λ_0 given in Lemma 5.10, we have

$$\langle \mathcal{L}^h v, v\rangle \;\le\; \Big(-\frac{1}{2}\lambda_0 + \mathcal{O}(\rho^2)\Big)\, \|v\|^2.$$

Proof. Since the minimal distance between the interfaces h_i is bounded from below by ε/ρ and the heteroclinic solution U goes exponentially fast to ± 1 (Proposition 5.3), we find $0 < \delta_\varepsilon < \frac{1}{2}\varepsilon/\rho$ such that $u^h = \pm 1 + \mathcal{O}(\exp)$ on $\mathcal{R} := [0,1] \setminus \bigcup B_{\delta_\varepsilon}(h_i)$. On the set $\mathcal{R}$ we have

$$\begin{aligned}
\langle \mathcal{L}^h v, v\rangle_{L^2(\mathcal{R})} \;&=\; -\varepsilon^2 \int_{\mathcal{R}} v_x^2 - \int_{\mathcal{R}} f'(\pm 1) v^2 + \mathcal{O}(\exp)\|v\|^2_{L^2(\mathcal{R})} \\
&\le\; -f'(\pm 1)\|v\|^2_{L^2(\mathcal{R})} + \mathcal{O}(\exp)\|v\|^2_{L^2(\mathcal{R})},
\end{aligned}$$

which is strictly negative as $f'(\pm 1) > 0$.

It remains to control the quadratic form on each $B_{\delta_\varepsilon}(h_i)$ and, without loss of generality, we may shift it to $h_i = 0$. Note that $u^h(x) = U(\frac{x-h_i}{\varepsilon}) + \mathcal{O}(\exp)$ on the set $B_{\delta_\varepsilon}(h_i)$ by Proposition 5.7. Defining $\tilde v(x) := v(\varepsilon x)$, we obtain for $h_i = 0$

$$\begin{aligned}
\langle \mathcal{L}^h v, v\rangle_{L^2(B_{\delta_\varepsilon})} \;&=\; -\varepsilon^2 \int_{-\delta_\varepsilon}^{\delta_\varepsilon} v'(x)^2\,\mathrm{d}x - \int_{-\delta_\varepsilon}^{\delta_\varepsilon} f'(U(\tfrac{x}{\varepsilon}))v(x)^2\,\mathrm{d}x + \mathcal{O}(\exp)\|v\|^2_{L^2(B_{\delta_\varepsilon})} \\
&=\; \varepsilon\left[-\varepsilon^2 \int_{-\delta_\varepsilon/\varepsilon}^{\delta_\varepsilon/\varepsilon} v'(\varepsilon y)^2\,\mathrm{d}y - \int_{-\delta_\varepsilon/\varepsilon}^{\delta_\varepsilon/\varepsilon} f'(U(y))v(\varepsilon y)^2\,\mathrm{d}y\right] + \mathcal{O}(\exp)\|v\|^2_{L^2(B_{\delta_\varepsilon})} \\
&=\; \varepsilon\left[-\int_{-\delta_\varepsilon/\varepsilon}^{\delta_\varepsilon/\varepsilon} \tilde v'(y)^2\,\mathrm{d}y - \int_{-\delta_\varepsilon/\varepsilon}^{\delta_\varepsilon/\varepsilon} f'(U(y))\tilde v(y)^2\,\mathrm{d}y\right] + \mathcal{O}(\exp)\|v\|^2_{L^2(B_{\delta_\varepsilon})} \\
&=\; \varepsilon\langle L\tilde v, \tilde v\rangle_{L^2(B_{\delta_\varepsilon/\varepsilon})} + \mathcal{O}(\exp)\|v\|^2_{L^2(B_{\delta_\varepsilon})}.
\end{aligned} \tag{5.11}$$

Here, L denotes the singular Sturm–Liouville operator defined by (5.7). After rescaling, we essentially have to bound the quadratic form $\langle L\tilde v, \tilde v\rangle$ on the interval $(-\delta_\varepsilon/\varepsilon, \delta_\varepsilon/\varepsilon) =: D_\varepsilon$, which is a set of length of order $\mathcal{O}(\rho^{-1}) = \mathcal{O}(\varepsilon^{-\kappa})$. To compare with the spectrum on the whole line, we define a cut-off function $\varphi \in C_c^\infty(D_\varepsilon)$ such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $\{f'(U) < C\}$ for some $\lambda_0 < C < \sup_{-1<x<1} f'(x)$. This is the only point where we need Assumption 5.11. As $|D_\varepsilon| = \mathcal{O}(\rho^{-1})$, we can also assume that uniformly $|\varphi_x| \le C\rho$ and $|\varphi_{xx}| \le C\rho^2$.

We obtain

$$
\begin{aligned}
\langle L\tilde v,\tilde v\rangle_{L^2(D_\varepsilon)} &= -\int_{D_\varepsilon}\tilde v_x^2-\int_{D_\varepsilon}f'(U)\,\tilde v^2\\
&= -\int_{D_\varepsilon}\tilde v_x^2\varphi^2-\int_{D_\varepsilon}(1-\varphi^2)\tilde v_x^2-\int_{D_\varepsilon}\varphi^2 f'(U)\tilde v^2-\int_{D_\varepsilon}(1-\varphi^2)f'(U)\tilde v^2\\
&\le -\int_{D_\varepsilon}\tilde v_x^2\varphi^2-\int_{D_\varepsilon}\varphi^2 f'(U)\tilde v^2-C\int_{D_\varepsilon}(1-\varphi^2)\tilde v^2\\
&= -\int_{D_\varepsilon}((\varphi\tilde v)_x)^2+\int_{D_\varepsilon}f'(U)(\varphi\tilde v)^2+2\int_{D_\varepsilon}\varphi_x\tilde v_x\varphi\tilde v+\int_{D_\varepsilon}\tilde v^2\varphi_x^2-C\int_{D_\varepsilon}(1-\varphi^2)\tilde v^2\\
&= \langle L\varphi\tilde v,\varphi\tilde v\rangle_{L^2(\mathbb{R})}-\int_{D_\varepsilon}\tilde v^2(\varphi\varphi_x)_x+\int_{D_\varepsilon}\tilde v^2\varphi_x^2-C\int_{D_\varepsilon}(1-\varphi^2)\tilde v^2\\
&\le -\frac12\lambda_0\int_{D_\varepsilon}\varphi^2\tilde v^2-\int_{D_\varepsilon}\tilde v^2(\varphi\varphi_x)_x+\int_{D_\varepsilon}\tilde v^2\varphi_x^2-C\int_{D_\varepsilon}(1-\varphi^2)\tilde v^2+\mathcal{O}(\exp)\\
&\le \left(-\frac12\lambda_0+\mathcal{O}(\rho^2)\right)\int_{D_\varepsilon}\tilde v^2-(C-\lambda_0)\int_{D_\varepsilon}(1-\varphi^2)\tilde v^2+\mathcal{O}(\exp)\\
&\le \left(-\frac12\lambda_0+\mathcal{O}(\rho^2)\right)\|\tilde v\|^2_{L^2(D_\varepsilon)}+\mathcal{O}(\exp). \qquad (5.12)
\end{aligned}
$$

Note that the exponentially small terms arise from the fact that $\varphi\tilde v$ is not exactly orthogonal to U', but the error is exponentially small as

$$
\begin{aligned}
\int_{D_\varepsilon}(\varphi\tilde v)\cdot U' &= \int_{D_\varepsilon}\tilde vU'-\int_{D_\varepsilon}(1-\varphi)\tilde vU'\\
&= \varepsilon^{-1}\int_{-\delta_\varepsilon}^{\delta_\varepsilon}v(x)U'(x/\varepsilon)-\int_{D_\varepsilon}(1-\varphi)\tilde vU' \;=\; \mathcal{O}(\exp)+\mathcal{O}(\exp)\int_{D_\varepsilon}(1-\varphi)\tilde v.
\end{aligned}
$$

Thus, by Lemma 2.10 we have

$$
\langle L\varphi\tilde v,\varphi\tilde v\rangle \;=\; -\int_{D_\varepsilon}((\varphi\tilde v)_x)^2+\int_{D_\varepsilon}f'(U)(\varphi\tilde v)^2 \;\le\; -\frac12\lambda_0\int_{D_\varepsilon}\varphi^2\tilde v^2+\mathcal{O}(\exp).
$$

Now, we observe that

$$
\|\tilde v\|^2_{L^2(D_\varepsilon)} \;=\; \int_{D_\varepsilon}v(\varepsilon x)^2\,\mathrm{d}x \;=\; \varepsilon^{-1}\int_{B_{\delta_\varepsilon}}v(y)^2\,\mathrm{d}y \;=\; \varepsilon^{-1}\|v\|^2_{L^2(B_{\delta_\varepsilon})},
$$

and therefore, combining (5.11) and (5.12) yields

$$
\begin{aligned}
\langle \mathcal{L}^h v,v\rangle_{L^2(B_{\delta_\varepsilon})} &= \varepsilon\langle L\tilde v,\tilde v\rangle_{L^2(B_{\delta_\varepsilon/\varepsilon})}+\mathcal{O}(\exp)\|v\|^2_{L^2(B_{\delta_\varepsilon})}\\
&\le \varepsilon\left(-\frac12\lambda_0+\mathcal{O}(\rho^2)\right)\|\tilde v\|^2_{L^2(D_\varepsilon)}+\mathcal{O}(\exp)\|v\|^2_{L^2(B_{\delta_\varepsilon})}\\
&= \left(-\frac12\lambda_0+\mathcal{O}(\rho^2)\right)\|v\|^2_{L^2(B_{\delta_\varepsilon})}+\mathcal{O}(\exp)\|v\|^2_{L^2(B_{\delta_\varepsilon})}. \qquad \square
\end{aligned}
$$

As a next step, we analyze the spectral gap in the mass conserving case. For this purpose, we denote by P be the projection of L^2 onto the linear subspace $L_0^2=\{f\in L^2 : \int_0^1 f(x)\,\mathrm{d}x=0\}$. Motivated by

$$
\langle P\mathcal{L}^hPv,v\rangle_{L^2} \;=\; \langle\mathcal{L}^hPv,Pv\rangle_{L^2} \;=\; \langle\mathcal{L}^hv,v\rangle_{L_0^2} \quad \text{for } v\in L_0^2,
$$

we observe that it is sufficient to consider the same operator $\mathcal{L}^h$ as for the classical Allen–Cahn equation, but restricted to the subspace L_0^2 of L^2 containing functions with mean zero. This constraint leads to a subspace of codimension 1,

$$
L_0^2 \;=\; \left\{v\in L^2 : \langle v,1\rangle \;=\; 0\right\} \;=\; 1^\perp \;=\; PL^2,
$$

and therefore, we need to control the quadratic form on this subspace. First, we will formulate the problem in general and only after that consider the special case for the mass conserving Allen–Cahn equation. The following theorem deals with establishing a spectral gap on a subspace of codimension 1. We show that, under a suitable angle condition, the Rayleigh quotient can be bounded from above. This yields a bound on the spectral gap.

Theorem 5.15 (Spectral gap on subspaces)**.**
Consider a self-adjoint operator $\mathcal{L}$ on a Hilbert space $\mathcal{H}$ with an orthonormal basis of eigenfunctions $\mathcal{L} f_k = \lambda_k f_k$ and assume that

$$\delta \geq \lambda_1, \ldots, \lambda_{N+1} \geq -\delta > -\lambda \geq \lambda_{N+2} \geq \ldots \tag{5.13}$$

for some $0 < \delta < \lambda$. For $u \in \mathcal{H}$, we define

$$u^\perp := \Big\{ f \in \mathcal{H} : \langle f, u \rangle = 0 \Big\} \quad \textit{and} \quad F_u := \frac{1}{\langle f_{N+1}, u \rangle} \sum_{i=1}^{N} \langle f_i, u \rangle f_i + f_{N+1}.$$

Then,

i) there exists an N-dimensional subspace $\mathcal{U}$ of $u^\perp$ such that

$$|\langle \mathcal{L} h, h \rangle| \leq \delta \|h\|^2 \quad \forall h \in \mathcal{U}.$$

ii) the condition $|\cos \sphericalangle(F_u, u)| \geq \sqrt{\delta/\lambda}$ implies that for $h \perp u, f_1, \ldots, f_{N+1}$

$$\frac{\langle \mathcal{L} h, h \rangle}{\|h\|^2} \leq \frac{\delta - \lambda \cos^2 \sphericalangle(F_u, u)}{\cos^2 \sphericalangle(F_u, u) + 1}.$$

Proof. First, we construct an N-dimensional subspace corresponding to the small eigenvalues in the interval $[-\delta, \delta]$. For $i = 1, \ldots, N$ define

$$g_i := f_i + c_i f_{N+1} \quad \text{with} \quad c_i := -\frac{\langle f_i, u \rangle}{\langle f_{N+1}, u \rangle}. \tag{5.14}$$

Obviously, we have $g_1, \ldots, g_N \in \operatorname{span}\{f_1, \ldots, f_{N+1}\}$ and $g_1, \ldots, g_N \perp u$ by the definition of the constant c_i. It is also straightforward to check that the functions g_i span an N-dimensional space. This yields directly

$$-\delta \|h\|^2 \leq \langle \mathcal{L} h, h \rangle \leq \delta \|h\|^2 \qquad \text{for } h \in \operatorname{span}\{g_1, \ldots, g_N\} =: \mathcal{U}.$$

Define $V := \operatorname{span}\{g_1, \ldots, g_N\}^\perp \cap u^\perp = \operatorname{span}\{u, g_1, \ldots, g_N\}^\perp$. For $h \in V$ we can then write

$$h = \sum_{i=1}^{N+1} \alpha_i f_i + r, \qquad \text{with} \quad r \perp f_i \quad \forall i = 1, \ldots, N+1. \tag{5.15}$$

We have $r, h \perp g_j$ for any $j = 1, \ldots, N$ and thereby

$$\sum_{i=1}^{N+1} \alpha_i \langle f_i, g_j \rangle = 0. \tag{5.16}$$

With (5.14) and $f_i \perp f_j$ for $i \neq j$, we easily compute that

$$\langle f_i, g_j \rangle_{i,j} = \begin{pmatrix} 1 & 0 & \cdots & 0 & c_1 \\ 0 & \ddots & \vdots & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 1 & c_N \end{pmatrix} \in \mathbb{R}^{N \times (N+1)}.$$

The kernel of this matrix is one-dimensional and spanned by a vector $\beta \in \mathbb{R}^{N+1}$ with $\beta_i = -c_i$ for $1 \leq i \leq N$ and $\beta_{N+1} = 1$. By (5.16), α lies in the kernel and we can rewrite (5.15) as

$$h \;=\; \gamma \sum_{i=1}^{N+1} \beta_i f_i + r \;=\; \gamma \cdot F_u + r, \quad \gamma \in \mathbb{R}.$$

Since $h \in V \subset u^\perp$, we have $0 = \langle h, u\rangle = \gamma\langle F_u, u\rangle + \langle r, u\rangle$. This implies immediately that

$$\gamma^2 \;=\; \frac{\langle r, u\rangle^2}{\langle F_u, u\rangle^2} \;\leq\; \frac{\|r\|^2\|u\|^2}{\langle F_u, u\rangle^2}.$$

Thus, we compute

$$\begin{aligned}\frac{\langle \mathcal{L}h, h\rangle}{\|h\|^2} &= \frac{\sum_{j=1}^{N+1}\alpha_j^2\langle \mathcal{L}f_j, f_j\rangle + \langle \mathcal{L}r, r\rangle}{\gamma^2\|F_u\|^2 + \|r\|^2} \leq \frac{\delta\sum\alpha_j^2 - \lambda\|r\|^2}{\gamma^2\|F_u\|^2 + \|r\|^2} \\ &\leq \frac{\delta\gamma^2\sum\beta_j^2 - \lambda\|r\|^2}{\gamma^2\|F_u\|^2 + \|r\|^2} = \frac{\delta\gamma^2\|F_u\|^2 - \lambda\|r\|^2}{\gamma^2\|F_u\|^2 + \|r\|^2} \leq \frac{\left(\delta\dfrac{\|u\|^2\|F_u\|^2}{\langle F_u, u\rangle^2} - \lambda\right)\|r\|^2}{\gamma^2\|F_u\|^2 + \|r\|^2}.\end{aligned} \tag{5.17}$$

At this point, we need the angle condition

$$\cos\sphericalangle(F_u, u) \;=\; \frac{\langle F_u, u\rangle}{\|F_u\|\|u\|} \;\geq\; \sqrt{\delta/\lambda}$$

to guarantee that the numerator is negative. Under this assumption, we can continue estimating (5.17) and derive

$$\frac{\langle \mathcal{L}h, h\rangle}{\|h\|^2} \;\leq\; \frac{\delta\dfrac{\|u\|^2\|F_u\|^2}{\langle F_u, u\rangle^2} - \lambda}{\dfrac{\|u\|^2\|F_u\|^2}{\langle F_u, u\rangle^2} + 1} \;=\; \frac{\delta - \lambda\cos^2\sphericalangle(F_u, u)}{1 + \cos^2\sphericalangle(F_u, u)}.$$

□

Finally, we can apply Theorem 5.15 to analyze the spectrum of the linearized mass conserving Allen–Cahn operator. Recall that it is crucial to have a good negative upper bound of the quadratic form orthogonal to the tangent space. We show that in this case the spectral gap is of order ε. This is quite different to the spectral gap for the Allen–Cahn equation (AC) without the mass constraint, which by Theorem 5.14 is of order $\mathcal{O}(1)$.

Theorem 5.16 (Spectral gap for (mAC))**.**
Let $v \in L_0^2(0,1)$ with $v \perp u_i^\xi$ for $i = 1, \ldots, N$. Then, we have

$$\langle \mathcal{L}^\xi v, v\rangle \;\leq\; (-\lambda_0\varepsilon + \mathcal{O}(\exp))\,\|v\|^2,$$

where λ_0 is the same constant as in Theorem 5.14.

Proof. In the notation of Theorem 5.15, we take $u = 1 \in L^2(0,1)$ such that $L_0^2 = \operatorname{span}\{u\}^\perp$, and $f_i = u_i^h$. Furthermore, we compute

$$\langle f_i, 1\rangle \;=\; \int_0^1 u_i^h(x)\,\mathrm{d}x \;=\; \mathcal{O}(\exp) + \int_0^1 U'(x\,;\,h_i, (-1)^{i+1})\,\mathrm{d}x \;=\; 2(-1)^i + \mathcal{O}(\exp).$$

With F_u defined as before in Theorem 5.15, this yields

$$\langle F_u, u\rangle \;=\; \frac{1}{\langle f_{N+1}, 1\rangle}\sum_{j=1}^{N+1}\langle f_j, 1\rangle^2 \;=\; 2(N+1)(-1)^{N+1} + \mathcal{O}(\exp).$$

We have $\|f_i\| = \mathcal{O}(\varepsilon^{-1/2})$ by Proposition 5.7 and thus $\|F_u\| = (N+1)\cdot\mathcal{O}(\varepsilon^{-1/2})$. Combined we obtain

$$\cos\sphericalangle(F_u, u) \;=\; \mathcal{O}(\varepsilon^{1/2}).$$

By Proposition 5.7, we have $\mathcal{L}(u^h) = \mathcal{O}(\exp)$. Differentiating with respect to h_i yields $\mathcal{L}^h u_i^h = \mathcal{O}(\exp)$ and hence, the first $N+1$ eigenvalues are exponentially small. This shows that we can choose $\delta = \mathcal{O}(\exp)$. Plugging this observation into Theorem 5.15 yields

$$\frac{\langle \mathcal{L}^\xi v, v\rangle}{\|v\|^2} \;\le\; \frac{\delta - \lambda_0\varepsilon}{1+\varepsilon} \;\le\; -\lambda_0\varepsilon + \mathcal{O}(\exp). \qquad \square$$

For the classical Allen–Cahn equation we established a spectral gap of order 1, whereas due to mass conservation the gap shrinks to $\mathcal{O}(\varepsilon)$ for (mAC). As we will see later in Theorems 5.42 and 5.44, this heavily influences the maximal radius and noise strength that we can treat in our stability analysis.

5.3 Analysis of the stochastic ODE

In this section, we give the stochastic ODEs governing the motion of the kinks for both cases. We show that for the non-massconserving Allen–Cahn equation the $N+1$ interfaces move—up to the time scale where a collision is likely to occur—independently according to Brownian motions projected onto the slow manifold. This is quite different to the mass-conserving case where (as one would expect) the dynamics is coupled through the mass constraint.

Before we analyze the stochastic ODEs for the interface motion, we have to introduce the new coordinate frame (Fermi coordinates, see Definition 2.2), in which we derive the differential equations for the shape variable h and the normal component v. Due to Theorems 5.14 and 5.16, we established good control of the quadratic form orthogonal to the tangent space $\mathcal{T}_h\mathcal{M}$, or $\mathcal{T}_\xi\mathcal{M}_\mu$, respectively. Therefore, we do not need any approximations thereof. This leads to the following definition of the Fermi coordinates.

Definition 5.17 (Fermi coordinates)**.**
Let $u(t)$ be the solution to (AC). For a fixed time $t > 0$, we define the pair of coordinates $(h(t), v(t)) \in \Omega_\rho \times L^2(0,1)$ such that

$$u(t) \;= u^{h(t)} + v(t), \quad v(t) \perp \mathcal{T}_{h(t)}\mathcal{M},$$

as the *Fermi coordinates* of $u(t)$.
In case of the mass conserving equation (mAC), the definition works analogously. One only has to replace the set of admissible interface positions Ω_ρ by the set $\mathcal{A}_\rho$ (given by (5.6)) and the slow manifold $\mathcal{M}$ by its mass conserving counterpart $\mathcal{M}_\mu$.

Later in Lemma 5.19 and Remark 5.20, we show that sufficiently close to the manifold $\mathcal{M}_{(\mu)}$ the Fermi coordinates are well-defined. For now, we assume that t is sufficiently small such that the coordinate system is well-defined. We start with deriving the effective equations for h and v. In Chapter 2.2, we computed that h is a diffusion process given by

$$dh \;=\; b(h,v)\,dt + \langle \sigma(h,v), dW\rangle, \tag{5.18}$$

where (cf. (2.11) and (2.12))

$$\sigma_r(h,v) \;=\; \sum_i A_{ri}^{-1} u_i^h \tag{5.19}$$

and

$$
\begin{aligned}
b_r(h,v) \;&= \sum_i A^{-1}_{ri}\langle u^h_i, \mathcal{L}(u^h+v)\rangle + \sum_i A^{-1}_{ri}\sum_j \langle u^h_{ij}, \mathcal{Q}\sigma_j\rangle \\
&\quad + \sum_{i,j,k} A^{-1}_{ri}\left[\frac{1}{2}\langle u^h_{ijk}, v\rangle - \langle u^h_{ij}, u^h_k\rangle - \frac{1}{2}\langle u^h_i, u^h_{jk}\rangle\right]\langle \mathcal{Q}\sigma_j, \sigma_k\rangle.
\end{aligned}
\tag{5.20}
$$

Note that, for the sake of simplicity, we expressed everything with respect to the coordinate h, although we introduced the coordinate ξ for the mass conserving equation. An essential point in the computation of the SDE is the invertibility of the matrix A (cf. Definition and Metatheorem 1) given by

$$A_{kj}(h,v) \;=\; \langle u^h_k, u^h_j\rangle - \langle u^h_{kj}, v\rangle.$$

5.3.1 Analysis of the stochastic ODE for (AC)

We start with the non-massconserving case. Here, we will see that A and its inverse are—up to exponentially small terms in $\|v\|$—diagonal matrices.

Lemma 5.18. *For $h \in \Omega_\rho$ consider the matrix $A \in \mathbb{R}^{(N+1)\times(N+1)}$ defined by*

$$A_{kj} \;:=\; \langle u^h_k, u^h_j\rangle - \langle u^h_{kj}, v\rangle.$$

We obtain

$$A_{kj} \;=\; \left[\mathcal{X}\varepsilon^{-1} + \mathcal{O}(\varepsilon^{-3/2})\|v\|\right]\delta_{kj} + \mathcal{O}(\exp)\|v\|.$$

Moreover, as long as $\|v\| < c\varepsilon^{1/2+m}$ for some $m > 0$, the inverse A^{-1} is given by

$$A^{-1} \;=\; \left[\mathcal{X}^{-1}\varepsilon + \mathcal{O}(\varepsilon^{1+m})\right]\mathrm{I}_{N+1} + \mathcal{O}(\exp),$$

where $\mathcal{X}$ is the constant given in Proposition 5.7.

Proof. We obtain $\langle u^h_k, u^h_j\rangle = \mathcal{X}\varepsilon^{-1}\delta_{kj} + \mathcal{O}(\exp)$ and $\|u^h_{kj}\| = \mathcal{O}(\varepsilon^{-3/2})\delta_{kj} + \mathcal{O}(\exp)$ by Proposition 5.7. Thus, the bound on A_{kj} follows directly by applying the Cauchy–Schwarz inequality. Using geometric series, this yields for $\|v\|$ sufficiently small

$$
\begin{aligned}
A^{-1} \;&=\; \left[\mathcal{X}\varepsilon^{-1} + \mathcal{O}(\varepsilon^{-3/2})\|v\|\right]^{-1}\mathrm{I}_{N+1} + \mathcal{O}(\exp) \\
&=\; \mathcal{X}^{-1}\varepsilon\left[1 + \mathcal{O}(\varepsilon^{-1/2})\|v\|\right]^{-1}\mathrm{I}_{N+1} + \mathcal{O}(\exp) \\
&=\; \left[\mathcal{X}^{-1}\varepsilon + \mathcal{O}(\varepsilon^{1+m})\right]\mathrm{I}_{N+1} + \mathcal{O}(\exp).
\end{aligned}
$$

□

Before we continue analyzing the stochastic ODE, let us first show that the coordinate frame around $\mathcal{M}$ given by Definition 5.17 is well-defined. We prove that, as long as the matrix A is invertible, i.e., $\|v\| < \varepsilon^{1/2+m}$, and the nonlinearity is bounded, i.e., $v \in L^4$, the coefficients b and σ defined by (5.20) and (5.19) are Lipschitz continuous with respect to h. Note that we will only compute the Lipschitz constant for σ explicitly, as we need it for the analysis of the stochastic ODE.

Lemma 5.19 (Lipschitz continuity of b and σ)**.**
Let $h, \bar{h} \in \Omega_\rho$. If $v \in L^4(0,1)$ with $\|v\|_{L^2} < \varepsilon^{1/2+m}$ for some $m > 0$, there exist constants $C > 0$ and $C_\varepsilon > 0$ (depending on ε and $\|v\|_{L^4}$) such that

$$\|\sigma(h,v) - \sigma(\bar{h},v)\| \;\le\; C\varepsilon^{-1/2}|h-\bar{h}| \quad \text{and} \quad \|b(h,v) - b(\bar{h},v)\| \;\le\; C_\varepsilon|h-\bar{h}|. \tag{5.21}$$

Proof. Note that in the following computation the pair (h, v) does not denote the Fermi coordinate defined in Definition 5.17 and therefore, v does not depend on h. We start with estimating the derivative of the inverse $A^{-1}(h, v)$. By construction of u^h, the matrix $A(h, v)$ is smooth in h and we compute

$$\partial_{h_k} A_{ij} \;=\; \frac{\partial(\langle u_i^h, u_j^h\rangle - \langle u_{ij}^h, v\rangle)}{\partial h_k} \;=\; \langle u_{ik}^h, u_j^h\rangle + \langle u_i^h, u_{jk}^h\rangle - \langle u_{ijk}^h, v\rangle,$$

which by Proposition 5.7 is exponentially small unless $i = j = k$. In the latter case, we have

$$\partial_{h_k} A_{kk} \;=\; 2\langle u_{kk}^h, u_k^h\rangle - \langle u_{kkk}^h, v\rangle \;=\; \mathcal{O}(\exp) + \mathcal{O}(\varepsilon^{-5/2}\|v\|).$$

By virtue of $\mathrm{D}_h A^{-1} = -A^{-1}(\mathrm{D}_h A)A^{-1}$ and $A^{-1} = \mathcal{O}(\varepsilon)$ (cf. Lemma 5.18), this yields

$$\mathrm{D}_h A^{-1} \;=\; \mathcal{O}(\varepsilon^{-1/2}\|v\|) \;=\; \mathcal{O}(\varepsilon^m).$$

Recall that $\sigma(h, v) = A^{-1} \cdot \partial_h u^h$. Differentiating with respect to h yields

$$\mathrm{D}_h\sigma \;=\; \mathrm{D}_h A^{-1}\partial_h u^h + A^{-1}\partial_h^2 u^h$$

and thus, by the previous bound on $\mathrm{D}_h A^{-1}$ and Proposition 5.7, $\|\mathrm{D}_h\sigma(h, v)\| \;=\; \mathcal{O}(\varepsilon^{-1/2})$. Since the set Ω_ρ of admissible interface positions is convex, we have

$$\sigma(h, v) - \sigma(\bar{h}, v) \;=\; \int_0^1 \mathrm{D}_h\sigma(\bar{h} + s(h - \bar{h}), v)\,\mathrm{d}s$$

and with that we easily obtain (5.21).

In order to derive the Lipschitz continuity of b, one can analogously verify that $b(h, v)$ is differentiable with respect to h and the derivative is bounded. Note that only here we need the condition $v \in L^4$ to control the nonlinearity $\langle \mathcal{N}^h(v), v\rangle$ appearing in the definition (5.20) of b. The careful analysis of the Lipschitz constant can be carried out after some lengthy calculation. We omit the details here. □

Remark 5.20. We can use the Lipschitz continuity of the coefficients to show that the Fermi coordinates given by Definition 5.17 are locally well defined (see Section 2.2.2). Let u be the unique solution to (AC). Since the multi-kink profiles u^h define smooth functions in h, we see that by Lemma 5.19 the maps $h \mapsto b(h, u - u^h)$ and $h \mapsto \sigma(h, u - u^h)$ are locally Lipschitz continuous. Thus, we find—as long as $h(t)$ lies in Ω_ρ and $u - u^h$ is sufficiently small—a unique solution $h(t)$ to (5.18) with v replaced by $u - u^h$. Defining $v(t) = u(t) - u^{h(t)}$ leads to a uniquely defined pair (h, v) that satisfies the Definition 5.17 of the Fermi coordinates. We refer to Lemma 2.8 for the general proof.

As the matrix A and its inverse are (up to exponentially small terms) diagonal matrices, we can show that the stochastic ODE in the non-massconserving case essentially decouples fully. We split equation (5.18) into its deterministic part and a remainder $\mathcal{A}$, where we collect all terms depending on stochastics, i.e., we write

$$dh_r \;=\; \sum_i A_{ri}^{-1}\langle u_i^h, \mathcal{L}(u^h + v)\rangle\,dt + d\mathcal{A}^{(r)}, \tag{5.22}$$

where by (5.19) and (5.20)

$$\begin{aligned}\sum_j A_{kj}\,d\mathcal{A}^{(j)} \;=\; &\sum_j \langle u_{kj}^h, \mathcal{Q}\sigma_j\rangle\,dt + \sum_{i,j}\left[\frac{1}{2}\langle u_{ijk}^h, v\rangle - \langle u_{kj}^h, u_i^h\rangle - \frac{1}{2}\langle u_k^h, u_{ij}^h\rangle\right]\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\,dt \\ &+ \langle u_k^h, dW\rangle.\end{aligned} \tag{5.23}$$

The following lemma deals with estimating the process $\mathcal{A}$ in terms of ε. We see that—up to lower order terms—the k-th component of $\mathcal{A}$ does only depend on the derivatives with respect to h_k and hence it decouples. Moreover, its dominating term is of order $\mathcal{O}(\eta_0)$.

Lemma 5.21. *As long as $\|v(t)\| < \varepsilon^{1/2+m}$ for some $m > 0$, we have*

$$d\mathcal{A}^{(k)} \;=\; A_{kk}^{-2}\langle u_{kk}^h, \mathcal{Q}u_k^h\rangle\, dt + A_{kk}^{-1}\langle u_k^h, dW\rangle + \mathcal{O}(\varepsilon^m\eta_0)\, dt + \langle \mathcal{O}_{L^2}(\varepsilon^{1/2+m}), dW\rangle.$$

Moreover, the dominating term can be estimated by

$$A_{kk}^{-2}\langle u_{kk}^h, \mathcal{Q}u_k^h\rangle\, dt + A_{kk}^{-1}\langle u_k^h, dW\rangle \;=\; \mathcal{O}(\eta_0)\, dt + \langle \mathcal{O}_{L^2}(\varepsilon^{1/2}), dW\rangle.$$

Proof. Lemma 5.18 and (5.19) imply directly that

$$\sigma_r(h,v) \;=\; \Big[\mathcal{X}^{-1}\varepsilon + \mathcal{O}(\varepsilon^{1+m})\Big]\, u_r^h + \mathcal{O}(\exp).$$

With $\|u_r^h\| = \mathcal{O}(\varepsilon^{-1/2})$ (cf. Proposition 5.7), this yields $\|\sigma_r\| = \mathcal{O}(\varepsilon^{1/2})$. The Cauchy–Schwarz inequality implies for the remaining terms of (5.23)

$$|\langle u_{kj}^h, \mathcal{Q}\sigma_j\rangle| \;\le\; \|u_{kj}^h\|\, \|\mathcal{Q}\|\, \|\sigma_j\| \;\le\; C\varepsilon^{-3/2}\eta_0\varepsilon^{1/2} \;=\; C\varepsilon^{-1}\eta_0$$

and

$$|\langle u_{ijk}^h, v\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle| \;\le\; C\varepsilon^{-5/2}\|v\|\eta_0\varepsilon^{1/2}\varepsilon^{1/2} \;=\; C\varepsilon^{-3/2}\eta_0\, \|v\|.$$

Moreover by Proposition 5.7, the terms involving inner products of first and second derivatives of u^h are exponentially small. Plugging these estimates into (5.23) yields

$$\sum_j A_{kj}\, d\mathcal{A}^{(j)} \;=\; \langle u_{kk}^h, \mathcal{Q}\sigma_k\rangle\, dt + \langle u_k^h, dW\rangle + \mathcal{O}(\varepsilon^{-1+m}\eta_0) + \mathcal{O}(\exp).$$

By using Lemma 5.18, we obtain

$$\begin{aligned} d\mathcal{A}^{(k)} \;&=\; A_{kk}^{-1}\Big[\langle u_{kk}^h, \mathcal{Q}\sigma_k\rangle\, dt + \langle u_k^h + \mathcal{O}(\exp), dW\rangle + \mathcal{O}(\varepsilon^{-1+m}\eta_0)\, dt + \mathcal{O}(\exp)\, dt\Big] \\ &=\; A_{kk}^{-2}\langle u_{kk}^h, \mathcal{Q}u_k^h\rangle\, dt + A_{kk}^{-1}\langle u_k^h + \mathcal{O}(\exp), dW\rangle + \mathcal{O}(\varepsilon^m\eta_0)\, dt + \mathcal{O}(\exp)\, dt. \end{aligned}$$

□

As a next step, we investigate the deterministic part. As we cannot control the nonlinearity in terms of the L^2-norm, we additionally assume smallness of v in L^4. In the stability result of Section 5.5, the maximal L^4-radius that we can treat is of order $\varepsilon^{1/4+m/2-\kappa}$ for small $\kappa > 0$.

Lemma 5.22. *Let $m > 0$ and $\kappa > 0$ be very small. For $h \in \Omega_\rho$ and $v \perp u_i^h, i = 1,\dots,N+1$, assume that $\|v\| < \varepsilon^{1/2+m}$ and $\|v\|_{L^4} < \varepsilon^{1/4+m/2-\kappa}$. Then, we have*

$$\sum_i A_{ri}^{-1}\langle u_i^h, \mathcal{L}(u^h+v)\rangle \;\le\; C\varepsilon^{2m+1-2\kappa}.$$

Proof. Expanding $\mathcal{L}$ yields $\mathcal{L}(u^h+v) = \mathcal{L}(u^h) + \mathcal{L}^h v + \mathcal{N}^h(v)$. We observe that $\mathcal{L}(u^h) = \mathcal{O}(\exp)$ by Proposition 5.7. Differentiating with respect to h_i yields $\mathcal{L}^h u_i^h = \mathcal{O}(\exp)$ and hence, $\langle \mathcal{L}^h v, u_i^h\rangle = \langle v, \mathcal{L}^h[u_i^h]\rangle = \mathcal{O}(\exp)$, since $\mathcal{L}^h$ is self-adjoint. The remaining nonlinear term is estimated by

$$\begin{aligned} \langle \mathcal{N}^h(v), u_i^h\rangle \;&=\; \int_0^1 3u^h u_i^h v^2 - u_i^h v^3\, \mathrm{d}x \le C\varepsilon^{-1}\Big[\|v\|^2 + \|v\|_{L^3}^3\Big] \\ &\le C\varepsilon^{-1}\Big[\|v\|^2 + \|v\|\|v\|_{L^4}^2\Big] \le C\varepsilon^{2m-2\kappa}, \end{aligned}$$

where we interpolated the L^3-term by Hölder's inequality. Applying Lemma 5.18 concludes the proof. □

We can finally show that, up to times of order $\mathcal{O}(\varepsilon\eta_0^{-1})$, the motion of the kinks is approximately given by the projection of the Wiener process onto the slow manifold $\mathcal{M}$ (see Section 2.2.3), that is, for $k = 1, \dots, N+1$

$$d\tilde{h}_k = \frac{1}{\|u_k^{\tilde{h}}\|^2}\langle u_k^{\tilde{h}}, \circ\, dW\rangle. \tag{5.24}$$

At times of order $\mathcal{O}(\varepsilon\eta_0^{-1})$ the droplet is expected to move by the magnitude of ε and hence, we treat the relevant time scale in our analysis, since we have to assure that the distance between two kinks is at least ε (cf. Remark 5.24). For a sufficiently large noise strength, the stochastic effects dominate the dynamics and hence, as expected, the approximation by the purely stochastic process is better for a larger noise strength η_0. In our main stability result (see Theorem 5.42) the maximal strength we can treat is of order ε^{1+2m}.

Theorem 5.23 (Approximation of the exact dynamics)**.**
Let $h(t)$ be a solution to (5.18) and $\tilde{h}(t)$ be a solution to (5.24). For $m > 0$ and small $\kappa > 0$, define the stopping time

$$\tau := \inf\Big\{t \geq 0 \,:\, h(t) \notin \Omega_\rho \quad or \quad \|v\| > \varepsilon^{1/2+m} \quad or \quad \|v\|_{L^4} > \varepsilon^{1/4+m/2-\kappa}\Big\}.$$

Then, for $T < c\varepsilon\eta_0^{-1} \wedge \tau$, we obtain

$$\mathbb{E}\sup_{0\leq t\leq T}|h(t) - \tilde{h}(t)| \leq C\varepsilon + C\varepsilon^{2m+2-2\kappa}\eta_0^{-1}.$$

Proof. For notational convenience, we define for $h, \tilde{h} \in \Omega_\rho$ the maps

$$\gamma_r(h) := \frac{u_r^h}{\|u_r^h\|^2} \quad \text{and} \quad \Delta(h,\tilde{h}) := \gamma_r(h) - \gamma_r(\tilde{h}).$$

By (5.18), (5.19), and Lemma 5.18, we derive for $t \leq T$

$$h_r(t) - \tilde{h}_r(t) \leq \int_0^t b_r(s) + I_r(\tilde{h}(s))\,\mathrm{d}s + \int_0^t \langle \Delta(h,\tilde{h}) + \mathcal{O}(\varepsilon^{1+m})u_r^h\,,\, \mathrm{d}W\rangle.$$

Here, $I(\tilde{h})$ collects all the terms that appear after a conversion of the Stratonovich SDE (5.24) into an Itô SDE. This is important, as we need the stochastic integral to be a martingale. These Itô-Stratonovich correction terms are essentially identical to the terms in (5.20), where we set $v = 0$ and replace the matrix $A(\tilde{h}, v)$ by $S_{kj}(\tilde{h}) = A(\tilde{h}, 0) = \langle u_k^{\tilde{h}}, u_j^{\tilde{h}}\rangle$. In more detail, one easily computes that

$$\begin{aligned}
I_r(\tilde{h}) &= \sum_i S_{ri}^{-1}\sum_j \langle u_{ij}^h, \mathcal{Q}\sigma_j(\tilde{h},0)\rangle + \sum_{i,j,k} S_{ri}^{-1}\Big[-\langle u_{ij}^h, u_k^h\rangle - \frac{1}{2}\langle u_i^h, u_{jk}^h\rangle\Big]\langle \mathcal{Q}\sigma_j(\tilde{h},0), \sigma_k(\tilde{h},0)\rangle \\
&= S_{rr}^{-2}\langle u_{rr}^{\tilde{h}}, \mathcal{Q}u_r^{\tilde{h}}\rangle + \mathcal{O}(\exp) = \mathcal{O}(\eta_0).
\end{aligned}$$

Here, we utilized that $S_{ri} = \langle u_r^{\tilde{h}}, u_i^{\tilde{h}}\rangle = \mathcal{X}^{-1}\varepsilon\delta_{ri} + \mathcal{O}(\exp)$ by Proposition 5.7 and, as $v = 0$, $\sigma_r(\tilde{h}, 0) = \sum S_{ri}^{-1}u_i^{\tilde{h}} = S_{rr}^{-1}u_r^{\tilde{h}} + \mathcal{O}(\exp)$. Moreover, the inner product of first derivatives with second derivatives of u^h is exponentially small due to Proposition 5.7.
In Lemmata 5.21 and 5.22, we established an L^∞-bound for b up to the stopping time τ, namely,

$$\sup_{0\leq t\leq\tau}|b| \leq c(\eta_0 + \varepsilon^{2m+1-2\kappa})$$

Combining this with the bound of the Ito-Stratonovich correction term I yields

$$\mathbb{E}\sup_{0\leq t\leq T}|h_r(t) - \tilde{h}_r(t)| \leq c(\eta_0 + \varepsilon^{2m+1-2\kappa})T + \mathbb{E}\sup_{0\leq t\leq T}\Big|\int_0^t \langle \Delta(h,\tilde{h}) + \mathcal{O}(\varepsilon^{1+m})u_r^h, \mathrm{d}W\rangle\Big|.$$

By Burkholder's inequality and Lipschitz continuity of γ with Lipschitz constant of order $\mathcal{O}(\varepsilon^{-1/2})$ (cf. Lemma 5.19), the martingale term is estimated by

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\Big|\int_0^t \langle \Delta(h,\tilde h)+\mathcal{O}(\varepsilon^{1+m})u_r^h, \mathrm{d}W\rangle\Big| \\
&\le C\mathbb{E}\left[\int_0^T \langle \Delta(h,\tilde h)+\mathcal{O}(\varepsilon^{1+m})\,u_r^h, \mathcal{Q}(\Delta(h,\tilde h)+\mathcal{O}(\varepsilon^{1+m})\,u_r^h)\rangle\,\mathrm{d}s\right]^{1/2} \\
&= C\mathbb{E}\left[\int_0^T \langle \Delta(h,\tilde h), \mathcal{Q}\Delta(h,\tilde h)\rangle + \mathcal{O}(\varepsilon^{1+m})\langle \mathcal{Q}u_r^h, \Delta(h,\tilde h)\rangle + \mathcal{O}(\varepsilon^{2+2m})\langle u_r^h, \mathcal{Q}u_r^h\rangle\right]^{1/2} \\
&\le C\mathbb{E}\left[\int_0^T \eta_0\|\Delta(h,\tilde h)\|^2 + \varepsilon^{1/2+m}\eta_0\|\Delta(h,\tilde h)\| + \varepsilon^{1+2m}\eta_0\right]^{1/2} \\
&\le C\mathbb{E}\left[\int_0^T \eta_0\|\Delta(h,\tilde h)\|^2 + \varepsilon^{1+2m}\eta_0\right]^{1/2} \\
&\le C\mathbb{E}\left[\int_0^T \eta_0\varepsilon^{-1}\|h(s)-\tilde h(s)\|^2\,\mathrm{d}s + \varepsilon^{1+2m}\eta_0 T\right]^{1/2} \\
&\le C\eta_0^{1/2}\varepsilon^{-1/2}T^{1/2}\,\mathbb{E}\sup_{0\le t\le T}\|h(t)-\tilde h(t)\|^2 + C\varepsilon^{1/2+m}\eta_0^{1/2}T^{1/2}.
\end{aligned}$$

With the assumption $T < c\varepsilon\eta_0^{-1}$, this implies

$$\mathbb{E}\sup_{0\le t\le T}|h(t)-\tilde h(t)| \le C\frac{(\eta_0+\varepsilon^{2m+1-2\kappa})T + \varepsilon^{1/2+m}\eta_0^{1/2}T^{1/2}}{1-\eta_0^{1/2}\varepsilon^{-1/2}T^{1/2}} \le C\varepsilon + C\varepsilon^{2m+2-2\kappa}\eta_0^{-1}. \quad \square$$

Remark 5.24. In the definition of admissible parameters Ω_ρ, we had to assume that the distance between two interfaces is bounded from below by $\mathcal{O}(\varepsilon^{1-})$. Since $\mathbb{E}\,h(\varepsilon\eta_0^{-1}) = h(0)+\mathcal{O}(\varepsilon)$, the interface positions $h(t)$ might have moved by order ε and thus, a collision of two interfaces can occur, which we cannot treat in our analysis. Therefore, up to the relevant time, the motion of the kinks behaves approximately like a Wiener process projected onto the slow manifold. After a breakdown of two interfaces, we can restart our analysis on a lower-dimensional slow manifold, where the number of kinks is reduced by two.

5.3.2 Analysis of the stochastic ODE for (mAC)

To conclude our study of the kink motion, we analyze the mass conserving Allen–Cahn equation. Recall that in this case, due to mass conservation, we reduced the parameter space Ω_ρ via $h_{N+1} = h_{N+1}(\xi)$ by one dimension and therefore obtain by chain rule and Lemma 5.9

$$u_k^\xi = u_k^h + (-1)^{N-k}u_{N+1}^h + \mathcal{O}(\exp). \tag{5.25}$$

Remark 5.25. Analogously to Remark 5.20, we can verify that the Fermi coordinates (ξ, v) around $\mathcal{M}_\mu$ are locally well-defined (cf. Definition 5.17). The crucial point is that the maps $\xi \mapsto b(\xi, u-u^\xi)$ and $\xi \mapsto \sigma(\xi, u-u^\xi)$ are sufficiently smooth. In Lemma 5.19, we proved the local Lipschitz continuity of the corresponding maps in the non-massconserving case. In fact, let us show that these maps are even smoother. By the expressions in (4.2) and (4.3), σ and b depend on ξ via various derivatives of u^ξ (up to the third order). Note that also the matrix A only depends on derivatives of u^ξ. Hence, if the profiles u^ξ are sufficiently smooth, the smoothness is directly inherited to the coefficients of the stochastic ODE and we then obtain a unique local solution to $d\xi = b(\xi, u-u^\xi)\,dt + \langle\sigma(\xi, u-u^\xi), dW\rangle$.

In our construction of the slow manifold, we summed up rescaled and translated solutions to the ODE $U'' - F'(U) = 0$, which is equivalent to solving

$$U' - \sqrt{2F(U)} = 0.$$

In the toy case $F(u) = \frac{1}{4}(u^2-1)^2$, one obtains the solution $\tanh(\frac{x}{\sqrt{2}})$, which is of course C^∞-smooth. Later in Section 5.6, we consider potentials F that are given by polynomials. Due to the smoothness of F in that case, we can also conclude that the heteroclinic U is C^∞. Thereby, we see that the multi-kink configuration u^ξ is sufficiently smooth with respect to ξ, which shows that the aforementioned maps are at least C^1-functions. For details on how to obtain the well-definedness of the Fermi coordinates, we refer to Remark 5.20 and Section 2.2.2.

Just like in the analysis of (AC), we first show the invertibiliy of the matrix A. To start with, we consider the submatrix $S_{kj} = \langle u_k^\xi, u_j^\xi\rangle$, which does not depend on v. Due to the coupling through the mass constraint, the matrix S and its inverse are no longer diagonal. As we will see, this has an impact on the stochastic ODE governing the motion of the kinks.

Lemma 5.26. *For $u^\xi \in \mathcal{M}_\mu$ and $j,k \in \{1,\dots,N\}$ we have*

$$S_{kj} \;=\; \langle u_k^\xi, u_j^\xi\rangle \;=\; \mathcal{X}\varepsilon^{-1}\left[\delta_{kj} + (-1)^{k+j}\right] + \mathcal{O}(\exp),$$

where $\mathcal{X}$ is the constant given in Proposition 5.7.

Proof. With Proposition 5.7 and the chain rule (5.25), we compute

$$\begin{aligned}\langle u_k^\xi, u_j^\xi\rangle \;&=\; \langle u_k^h + (-1)^{N-k}u_{N+1}^h, u_j^h + (-1)^{N-j}u_{N+1}^h\rangle \\ &=\; \|u_k^h\|^2\,\delta_{jk} + (-1)^{k+j}\|u_{N+1}^h\|^2 + \mathcal{O}(\exp) \;=\; \mathcal{X}\varepsilon^{-1}\left[\delta_{kj} + (-1)^{k+j}\right] + \mathcal{O}(\exp). \qquad \square\end{aligned}$$

With the structure of the matrix at hand, we can easily invert S.

Lemma 5.27. *Let $u^\xi \in \mathcal{M}_\mu$. The matrix S is invertible with*

$$S_{kj}^{-1} \;=\; \varepsilon/\mathcal{X}\left[\delta_{kj} + \frac{1}{N+1}(-1)^{k+j+1}\right] + \mathcal{O}(\exp).$$

Proof. We have (ignoring exponentially small terms)

$$\begin{aligned}\sum_{j=1}^N S_{kj}S_{jl}^{-1} \;&=\; \sum_{j=1}^N \left[\delta_{kj} + (-1)^{k+j}\right]\left[\delta_{jl} + \frac{1}{N+1}(-1)^{j+l+1}\right] \\ &=\; \delta_{jl} + \frac{1}{N+1}(-1)^{k+l+1} + (-1)^{k+l} + \frac{N}{N+1}(-1)^{k+l+1} \;=\; \delta_{kl}. \qquad \square\end{aligned}$$

Finally, we show that—as long as $\|v\|$ stays sufficiently small—the full matrix $A(\xi, v)$ given by Definition 1 is invertible. With that, the coefficients of the Itô diffusion (5.18) (with h replaced by ξ) are well-defined and we can continue to study the dynamics of kinks for the mass conserving Allen–Cahn equation in more detail.

Lemma 5.28. *Consider the matrix $A_{kj}(\xi, v) = S_{kj} - \langle u_{kj}^\xi, v\rangle$, where S is given by Lemma 5.26. Then, as long as $\|v\| < \varepsilon^{1/2+m}$ for some $m > 0$, A is invertible with*

$$A^{-1} \;=\; S^{-1} + \mathcal{O}(\varepsilon^{m+1}).$$

Proof. For a small perturbation $S(v)$, given by $S_{kj}(v) = \langle u^\xi_{kj}, v\rangle$, of the matrix S we compute via geometric series

$$\begin{aligned} A^{-1} &= [S - S(v)]^{-1} = \left[\mathrm{I}_N - S^{-1}S(v)\right]^{-1} S^{-1} \\ &= \sum_{j\in\mathbb{N}} \left[S^{-1}S(v)\right]^j S^{-1} = S^{-1} + \sum_{j=1}^{\infty} \left[S^{-1}S(v)\right]^j S^{-1} = S^{-1} + \mathcal{O}(\varepsilon^{m+1}), \end{aligned}$$

where we used that $S(v) = \mathcal{O}(\varepsilon^{-3/2}\|v\|)$ and the sum converges for $\|v\| < \varepsilon^{1/2+m}$. □

We continue with estimating the deterministic part of (5.18). Similarly to Lemma 5.22, we have to assume smallness of the normal component v in L^2 and L^4 to control the nonlinearity. In the following lemma, we consider the radii for which we show stochastic stability later in Sections 5.4 and 5.5.

Lemma 5.29. *Let $m > 0$, $\xi \in \mathcal{A}_\rho$, and $v \perp u^\xi_i$ for $i = 1, \dots, N$. Also, assume that $\|v\| < \varepsilon^{3/2+m}$ and $\|v\|_{L^4} < \varepsilon^{3/4+m/2-\kappa}$. Then, we obtain*

$$\langle u^\xi_i, \mathcal{L}(u^\xi + v)\rangle \le C\varepsilon^{2+2m-2\kappa}.$$

Proof. We follow the proof of Lemma 5.22. Only for the nonlinearity we have to take the different radii into account. By Hölder's inequality we obtain

$$\langle \mathcal{N}^\xi(v), u^\xi_i\rangle \le C\varepsilon^{-1}\left[\|v\|^2 + \|v\|^3_{L^3}\right] \le C\varepsilon^{-1}\left[\|v\|^2 + \|v\|\|v\|^2_{L^4}\right] \le C\varepsilon^{2+2m-2\kappa}. \qquad □$$

In order to analyze the SDE governing the motion of kinks, it is more convenient to rewrite (5.18) in the Stratonovich sense. By leaving out Itô corrections, Lemmata 5.27 and 5.28 imply

$$\begin{aligned} d\xi_r &= \sum_i A^{-1}_{ri}\langle \mathcal{L}(u^\xi + v), u^\xi_i\rangle\, dt + \sum_i A^{-1}_{ri}\langle u^\xi_i, \circ\, dW\rangle \\ &= \sum_i S^{-1}_{ri}\langle \mathcal{L}(u^\xi + v), u^\xi_i\rangle\, dt + \sum_i S^{-1}_{ri}\langle u^\xi_i, \circ\, dW\rangle + \mathcal{O}(\varepsilon^{4+3m})\, dt + \langle \mathcal{O}_{L^2}(\varepsilon^{5/2+m}), \circ\, dW\rangle \\ &= \mathcal{X}^{-1}\varepsilon\, \langle \mathcal{L}(u^\xi + v), u^\xi_r\rangle\, dt + \mathcal{X}^{-1}\varepsilon\, \langle u^\xi_r, \circ\, dW\rangle \\ &\quad + \frac{(-1)^r\varepsilon}{\mathcal{X}(N+1)}\langle \mathcal{L}(u^\xi + v), \sum_{i=1}^N (-1)^{i+1}u^\xi_i\rangle\, dt + \frac{(-1)^r\varepsilon}{\mathcal{X}(N+1)}\langle \sum_{i=1}^N (-1)^{i+1}u^\xi_i, \circ\, dW\rangle \\ &\quad + \mathcal{O}(\varepsilon^{4+3m})\, dt + \langle \mathcal{O}_{L^2}(\varepsilon^{5/2+m}), \circ\, dW\rangle. \end{aligned}$$

The first two summands (depending only on u^ξ_r) are similar to the non-massconserving case, but—due to the mass constraint—we obtain additional terms, which do not only depend on the position ξ_r but rather on all positions $(\xi_1, \dots, \xi_N)$. To give a better understanding of this equation—especially of the additional terms—let us express it in the original h-coordinates. Recall that by chain rule $u^\xi_i = u^h_i + (-1)^{N-i}u^h_{N+1} + \mathcal{O}(\exp)$. Thus we compute (ignoring exponentially small terms)

$$\begin{aligned} u^\xi_r + \frac{(-1)^r}{N+1}\sum_{i=1}^N (-1)^{i+1}u^\xi_i &= u^h_r + (-1)^{N-r}u^h_{N+1} + \frac{(-1)^{r+N+1}N}{N+1}u^h_{N+1} + \frac{(-1)^r}{N+1}\sum_{i=1}^N (-1)^{i+1}u^h_i \\ &= u^h_r + (-1)^r u^h_{N+1}\left[\frac{(-1)^{N+1}N}{N+1} - (-1)^{N+1}\right] + \frac{(-1)^r}{N+1}\sum_{i=1}^N (-1)^{i+1}u^h_i \\ &= u^h_r + \frac{(-1)^r}{N+1}(-1)^N u^h_{N+1} + \frac{(-1)^r}{N+1}\sum_{i=1}^N (-1)^{i+1}u^h_i = u^h_r + \frac{(-1)^r}{N+1}\sum_{i=1}^{N+1} (-1)^{i+1}u^h_i. \end{aligned} \tag{5.26}$$

Plugging this into the Stratonovich SDE yields

$$\begin{aligned} d\xi_r \;=\; & \|u_r^h\|^{-2}\,\langle \mathcal{L}(u^h+v),u_r^h\rangle\,dt + \|u_r^h\|^{-2}\,\langle u_r^h,\circ\,dW\rangle \\ & + \frac{(-1)^r}{(N+1)}\sum_{i=1}^{N+1}(-1)^{i+1}\Big[\|u_i^h\|^{-2}\langle\mathcal{L}(u^h+v),u_i^h\rangle + \frac{(-1)^r}{(N+1)}\|u_i^h\|^{-2}\langle u_i^h,\circ\,dW\rangle\Big] \\ & + \mathcal{O}(\varepsilon^{4+3m})\,dt + \langle \mathcal{O}_{L^2}(\varepsilon^{5/2+m}),\circ\,dW\rangle. \end{aligned}$$

We observe that all the terms appearing in this formula are up to an exponentially small error the right-hand side of the equation for dh (see (5.22) and (5.23) with $A(h,v)$ a diagonal matrix). Thus, we have

$$d\xi_r \;\approx\; dh_r + \frac{(-1)^r}{(N+1)}\sum_{i=1}^{N+1}(-1)^{i+1}dh_i. \tag{5.27}$$

Therefore, the kink motion for the mass conserving Allen–Cahn equation is approximately given by the independent motion of the position h_r, which is moving according to the non-massconserving case, plus a weighted motion of all interface positions $(h_1,\ldots,h_{N+1})$ that guarantees the conservation of mass.

Remark 5.30. In Theorem 5.23, we proved that up to times of order $\varepsilon\eta_0^{-1}$ the interface positions $h(t)$ behave approximately like the projection of the Wiener process onto the slow manifold $\mathcal{M}$. Using that $dh_r \approx \|u_r^h\|^{-2}\langle u_r^h,\circ\,dW\rangle$ and plugging this into (5.27), we obtain heuristically

$$d\xi_r \;\approx\; \sum_i S_{ri}^{-1}\langle u_i^\xi,\circ\,dW\rangle, \tag{5.28}$$

where we essentially used the identity (5.26). Since the matrix S is given by $S_{ri}=\langle u_r^\xi,u_i^\xi\rangle$, we expect that also the dynamics for the mass conserving Allen–Cahn equation behaves approximately like the projection of the Wiener process onto $\mathcal{M}_\mu$. Analogously to Theorem 5.23, we could make this rigorous and estimate the error for a given time scale. Opposed to the previous analysis of (AC), we cannot quite reach a good error estimate up the relevant time scale of order $\mathcal{O}(\varepsilon\eta_0^{-1})$, which corresponds to the time that a kink is likely to move by the order of ε (see Remark 5.24). Basically, this deficiency stems from the worse spectral gap in Theorem 5.16, which leads to a smaller maximal noise strength that we can treat in our stability analysis. See Theorem 5.44, where we can allow only for $\eta_0\le\varepsilon^{4+2m-\kappa}$, and Theorem 2.14 for the interplay between the spectral gap and the noise strength. For a reasonable result, we need that the error, which is linear in the time scale T_ε, is smaller than the magnitude of the process ξ, which grows like $T_\varepsilon^{1/2}$. Note that we encountered this problem also in the analysis of the Cahn–Hilliard equation in higher space dimension (cf. Theorem 3.19 and the comment thereafter). In the case of the mass-conserving Allen–Cahn equation, we expect the following result to hold true but omit the details.

Conjecture 5.31. Let $\xi(t)$ be the solution to (5.18) with b and σ given by (5.20) and (5.19) and h replaced by ξ. Furthermore, let $\bar\xi(t)$ be the projection of the Wiener process W onto the mass conserving manifold $\mathcal{M}_\mu$ given by (5.28). For $m>0$ and small $\kappa>0$, define the exit time

$$\tau := \inf\Big\{t\ge 0 \,:\, \xi\notin\mathcal{A}_\rho \quad\text{or}\quad \|v(t)\|>\varepsilon^{3/2+m} \quad\text{or}\quad \|v(t)\|_{L^4}>\varepsilon^{3/4+m/2-\kappa}\Big\}$$

Then, for $T_\varepsilon\le c\varepsilon\eta_0^{-1}\wedge\tau$, we obtain

$$\mathbb{E}\sup_{0\le t\le T_\varepsilon}|\xi(t)-\bar\xi(t)| \;\le\; c\Big[\eta_0+\varepsilon^{3+2m-2\kappa}\Big]\,T_\varepsilon.$$

Remark 5.32. In the proof of the analogous result in Theorem 5.23, it was crucial to explicitly know the Lipschitz constant of the map $\xi \mapsto \sigma(\xi, v)$ provided v is sufficiently small. In establishing the Lipschitz continuity (Lemma 5.19), we relied on the convexity of the set of admissible interface positions. While this is straightforward for the set Ω_ρ, this is not quite true in the mass conserving case, where the set of admissible positions is given by

$$\mathcal{A}_\rho \coloneqq \Big\{(\xi, h_{N+1}(\xi)) \in \Omega_\rho \,:\, \xi \in [0,1]^N\Big\}.$$

By Lemma 5.9, h_{N+1} is explicitly given by

$$h_{N+1}(h_1, \dots, h_N) \;=\; \sum_{i=1}^{N} (-1)^{N-i} h_i + c(\mu) + \mathcal{O}(\exp),$$

where we have to introduce a constant $c(\mu)$ depending only on the mass μ. With this expression, one readily computes that $h_{N+1}(\xi+\lambda(\xi-\bar{\xi})) = h_{N+1}(\xi)+\lambda h_{N+1}(\xi-\bar{\xi})+\mathcal{O}(\exp)$ for any $\xi, \bar{\xi} \in \mathbb{R}^N$ and $\lambda \in (0,1)$. Combined with the convexity of Ω_ρ, this shows that the set $\mathcal{A}_\rho$ is not exactly convex, but the error is exponentially small. We obtain the following result:

$$h, \bar{h} \in \mathcal{A}_\rho \implies \lambda h + (1-\lambda)\bar{h} \in \mathcal{A}_{2\rho} \quad \forall \lambda \in (0,1).$$

With this property at hand, we expect to gain the Lipschitz constant in the mass conserving case. For some of the technical details, see the proof of Lemma 5.19.

5.4 Stability in L^2

We discuss stochastic stability in L^2, both for (AC) and (mAC). In our analysis, we follow the guideline of Section 2.3 very closely. Note that this is not sufficient for the analysis of the SDE, where we additionally assumed that v is small in L^4. Hence, we extend the stability result to L^4 afterwards in Section 5.5. Recall that the motion orthogonal to the slow manifold is given by (see (2.13))

$$dv \;=\; \Big[\mathcal{L}(u^h) + \mathcal{L}^h v + \mathcal{N}^h(v)\Big]\, dt + dW - \sum_j u_j^h dh_j - \frac{1}{2} \sum_{i,j} u_{ij}^h \langle \mathcal{Q}\sigma_i, \sigma_j \rangle \, dt.$$

Our aim is first to establish a bound of $d\|v\|^2 = 2\langle v, dv\rangle + \langle dv, dv\rangle$ of the same type as in Theorem 2.13, and then apply the main stability result of Theorem 2.14.

5.4.1 L^2-Stability for (AC)

We start with the analysis of (AC) without mass conservation. Crucial for establishing stochastic stability is the following theorem, which relies on the spectral gap derived in Theorem 5.14. As long as the L^2-norm of the normal component v stays sufficiently small, the nonlinear term does not destroy the spectral estimate.

Theorem 5.33. *Let $u^h \in \mathcal{M}$ and $v \perp u_i^h$, $i = 1, \dots, N+1$. Assume that $\|v\| < \varepsilon^{1/2+m}$ for some $m > 0$. Then, for λ_0 the constant given in the spectral bound of Theorem 5.14, we obtain*

$$\langle \mathcal{L}^h v + \mathcal{N}^h(v), v\rangle \;\leq\; -\frac{1}{2}\lambda_0 \,\|v\|^2.$$

Proof. Let $v \perp u_i^h \;\; \forall i = 1, \dots, N+1$. By the main spectral result of Theorem 5.14, we have

$$\langle \mathcal{L}^h v, v\rangle \;\leq\; -\lambda_0 \|v\|^2.$$

Therefore, for $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 = 1$, we compute

$$\begin{aligned}\langle \mathcal{L}^h v, v\rangle &\leq -\gamma_1\lambda_0\|v\|^2 + \gamma_2\varepsilon^2 \int_0^1 v_{xx}\, v \,\mathrm{d}x + \gamma_2 \int_0^1 f'(u^h)v^2 \,\mathrm{d}x \\ &\leq -\gamma_1\lambda_0\|v\|^2 - \varepsilon^2\gamma_2\|v_x\|^2 + \gamma_2\|f'(u^h)\|_{L^\infty}\|v\|^2.\end{aligned} \tag{5.29}$$

By Gagliardo–Nirenberg and Young‘s inequality we obtain

$$\begin{aligned}\langle \mathcal{N}^h(v), v\rangle &= \int_0^1 3(u^h)^2v^3 - v^4 \leq 3\|v\|_{L^3}^3 \leq C\|v_x\|^{1/2}\|v\|^{5/2} \\ &\leq \varepsilon^2\gamma_2\|v_x\|^2 + C\varepsilon^{-2/3}\gamma_2^{-1/3}\|v\|^{4/3}\|v\|^2,\end{aligned} \tag{5.30}$$

where we interpolated the L^3-norm between H^1 and L^2. Combining (5.29) and (5.30) yields

$$\begin{aligned}\langle \mathcal{L}^h v + \mathcal{N}^h(v), v\rangle &\leq -\gamma_1\lambda_0\|v\|^2 + \left[\gamma_2\|f'(u^h)\|_{L^\infty} + C\varepsilon^{-2/3}\gamma_2^{-1/3}\|v\|^{4/3}\right]\|v\|^2 \\ &= \left[-\lambda_0 + \gamma_2\lambda_0 + \gamma_2\|f'(u^h)\|_{L^\infty} + C\varepsilon^{-2/3}\gamma_2^{-1/3}\|v\|^{4/3}\right]\|v\|^2.\end{aligned}$$

Fixing $\gamma_2 = \varepsilon^m$, we obtain for $\|v\| < \varepsilon^{1/2+m}$

$$\langle \mathcal{L}^h v + \mathcal{N}^h(v), v\rangle \leq \left[-\lambda_0 + \varepsilon^m\left(\lambda_0 + \|f'(u^h)\|_\infty\right) + C\varepsilon^m\right]\|v\|^2. \qquad \square$$

As a next step, we need to analyze the remaining terms of $d\|v\|^2$. We show that, provided $\|v\|$ is sufficiently small, they are of order $\mathcal{O}(\eta_0)$.

Lemma 5.34. *Under the same assumptions as in Theorem 5.33, we obtain*

$$\langle \mathcal{L}(u^h), v\rangle\, dt - \frac{1}{2}\sum_{i,j}\langle u^h_{ij}, v\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\, dt + \langle dv, dv\rangle = \mathcal{O}(\eta_0)\, dt.$$

Proof. We have $\mathcal{L}(u^h) = \mathcal{O}(\exp)$ and, as $\|v\| < \varepsilon^{1/2+m}$,

$$\langle u^h_{ij}, v\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle \leq c\varepsilon^{-3/2}\|v\|\eta_0\varepsilon^{1/2}\varepsilon^{1/2} = \mathcal{O}(\varepsilon^m\eta_0).$$

For the Itô correction term $\langle dv, dv\rangle$ we see that

$$\langle dv, dv\rangle = \eta_0\, dt + \sum_{i,j}\left[\langle u^h_i, u^h_j\rangle - 2\langle \mathcal{Q}u^h_j, \sigma_j\rangle\right] dt = \mathcal{O}(\eta_0)\, dt.$$

Here, we utilized that $\|u^h_i\| = \mathcal{O}(\varepsilon^{-1/2})$, $\|u^h_{ij}\| = \mathcal{O}(\varepsilon^{-3/2})$ (Proposition 5.7), and $\|\sigma_i\| = \mathcal{O}(\varepsilon^{1/2})$ (Lemma 5.21). $\square$

Combining the estimates of Theorem 5.33 and Lemma 5.34, we fully estimated the stochastic differential $d\|v\|^2$. This provides us with the following result, which is essential for proving stability in L^2 (see Theorem 2.13).

Corollary 5.35. *Let $u^h \in \mathcal{M}$. If $v \perp u^h_i$ for $i = 1, \dots, N+1$ and $\|v\| < \varepsilon^{1/2+m}$ for some $m > 0$, we obtain*

$$d\|v\|^2 \leq \left[-\frac{1}{2}\lambda_0\|v\|^2 + \mathcal{O}(\eta_0)\right] dt + 2\langle v, dW\rangle.$$

We can finally show that the L^2-norm of v stays small for very long times under small stochastic perturbations. Since the following stability results can only hold as long as $h(t) \in \Omega_\rho$, we define the first exit time from the open set Ω_ρ by

$$\tau_0 := \left\{t \geq 0 \,:\, h(t) \notin \Omega_\rho\right\}. \tag{5.31}$$

Note that we have seen in Remark 5.24 that at times of order $\mathcal{O}(\varepsilon\eta_0^{-1})$ the interface positions are likely to move by the magnitude of ε and thus exit the set of admissible positions Ω_ρ. This suggests that—if stability holds— the exit time τ_0 is with high probability of order $\mathcal{O}(\varepsilon\eta_0^{-1})$.

Theorem 5.36 (L^2-Stability for (AC))**.**
For $m > 0$ define the stopping time

$$\tau^* := \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0] : \|v(t)\| > \varepsilon^{1/2+m}\right\},$$

where the deterministic cut-off satisfies $T_\varepsilon = \varepsilon^{-M}$ for fixed large $M > 0$ and τ_0 is given by (5.31). Also, assume that for some $\nu \in (0, 1)$

$$\|v(0)\| \leq \nu\varepsilon^{1/2+m} \quad \text{and} \quad \eta_0 \leq \varepsilon^{1+2m}.$$

Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as ε tends to zero.

Proof. The statement follows directly by combining the estimate of Corollary 5.35 with the general stability result of Theorem 2.14. □

5.4.2 L^2-Stability for (mAC)

As a next step, we study the L^2-stability for the mass conserving Allen–Cahn equation. For the most part, we can rely on the results of the preceding section. The main difference lies in the fact that the spectral gap is only of order $\mathcal{O}(\varepsilon)$ opposed to an $\mathcal{O}(1)$-gap in the previous case. For that reason, we need to adept the proofs slightly.

Theorem 5.37. *Let $u^\xi \in \mathcal{M}_\mu$ and $v \perp u_i^\xi$, $i = 1, \dots, N$. Then, as long as $\|v\| < \varepsilon^{3/2+m}$ for some $m > 0$, we have*

$$\langle \mathcal{L}^\xi v + \mathcal{N}^\xi(v), v\rangle \leq -\frac{1}{2}\varepsilon\lambda_0 \|v\|^2,$$

with λ_0 independent of ε given by Theorem 5.14.

Proof. We can follow the proof of Theorem 5.33 closely, but have to take into account that by Theorem 5.16 the spectral gap is of order $\mathcal{O}(\varepsilon)$. Moreover, the additional term in the equation vanishes, as v is orthogonal to constants. Therefore, using the adapted version of (5.29) and the same interpolation as in (5.30), we derive

$$\langle \mathcal{L}^\xi v + \mathcal{N}^\xi(v), v\rangle \leq \left[-\lambda_0\varepsilon + \gamma_2\lambda_0\varepsilon + \gamma_2\|f'(u^h)\|_{L^\infty} + C\varepsilon^{-2/3}\gamma_2^{-1/3}\|v\|^{4/3}\right]\|v\|^2.$$

In order to absorb the term involving the L^∞-bound of the derivative of f into the negative term of order $\mathcal{O}(\varepsilon)$, we choose $\gamma_2 < \frac{1}{4\|f'(u^h)\|_{L^\infty}}\varepsilon$. This choice implies for the last term

$$C\varepsilon^{-2/3}\gamma_2^{-1/3}\|v\|^{4/3} < \frac{1}{4}\lambda_0\varepsilon \iff \|v\| < c\varepsilon^{3/2}$$

for some sufficiently small constant $c > 0$. Hence, as long as $\|v\| < \varepsilon^{3/2+m}$ for some $m > 0$, we obtain

$$\langle \mathcal{L}^\xi v + \mathcal{N}^\xi(v), v\rangle \leq -\frac{1}{2}\varepsilon\lambda_0\|v\|^2.$$ □

In the same way as in Lemma 5.34 (just exchange h with ξ and use the same estimates), we see that the remaining terms can be bounded by $\mathcal{O}(\eta_0)$. This leads to the following estimate for $d\|v\|^2$.

Corollary 5.38. *Let $u^\xi \in \mathcal{M}_\mu$. If $v \perp u_i^\xi$, $i = 1, \dots, N$, and $\|v\| < \varepsilon^{3/2+m}$ for some $m > 0$, we have*

$$d\|v\|^2 \le \left[-\frac{1}{2}\lambda_0\varepsilon\|v\|^2 + \mathcal{O}(\eta_0)\right] dt + 2\langle v, dW\rangle.$$

With Corollary 5.38 at hand, we can establish a stability result in the mass conserving case. Due to the worse spectral gap, the maximal noise strength that we can treat gets smaller.

Theorem 5.39 (L^2-Stability for (mAC))**.**
For $m > 0$ define the stopping time

$$\tau^* := \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0] \,:\, \|v(t)\| > \varepsilon^{3/2+m}\right\},$$

where $T_\varepsilon = \varepsilon^{-M}$ for fixed large $M > 0$ and τ_0 denotes the first exit time from $\mathcal{A}_\rho$. Also, assume that for some $\nu \in (0,1)$

$$\|v(0)\| \le \nu\varepsilon^{3/2+m} \quad \text{and} \quad \eta_0 \le \varepsilon^{4+2m}.$$

Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as ε tends to zero.

Proof. Once again, the assertion follows directly by combining the estimate of Corollary 5.38 with the general stability result of Theorem 2.14. □

5.5 Stability in L^4

For controlling the stochastic ODE of the interface positions, we need to establish bounds on the nonlinear term

$$\langle \mathcal{N}^h(v), u_i^h\rangle = \int_0^1 \left(3u^h v^2 - v^3\right) u_i^h \,\mathrm{d}x.$$

Since smallness in L^2 is not sufficient, we will prove that the L^4-norm of v stays small for very long times with high probability. In our analysis, we rely on the results of the preceding section. There, we established stochastic stability in L^2 and hence, all constants which appear in the following computations may depend on $\|v\|_{L^2}$ which—provided the assumptions of Theorem 5.36 hold true—is smaller than $\varepsilon^{1/2+m}$ for polynomial times in ε^{-1}.

5.5.1 L^4-Stability for (AC)

We begin with the classical Allen–Cahn equation (AC) without mass conservation. By the Itô formula we have

$$\frac{1}{4}d\|v\|_{L^4}^4 = \langle v^3, dv\rangle + 3\int_0^1 v^2 (dv)^2 \,\mathrm{d}x.$$

Again, recall that by (2.13)

$$dv = \left[\mathcal{L}(u^h) + \mathcal{L}^h v + \mathcal{N}^h(v)\right] dt + dW - \sum_j u_j^h dh_j - \frac{1}{2}\sum_{i,j} u_{ij}^h \langle \mathcal{Q}\sigma_i, \sigma_j\rangle \,dt.$$

First, let us estimate the Itô correction term $\int_0^1 v^2 (dv)^2 \,\mathrm{d}x$.

Lemma 5.40. *Let $h \in \Omega_\rho$. We obtain*

$$\int_0^1 v^2 (dv)^2 \,\mathrm{d}x = \mathcal{O}(\eta_0)\|v\|^2 \,dt.$$

Proof. Using the relation for dv we see that

$$\begin{aligned}\operatorname{trace}(\mathcal{Q})\int_0^1 v^2\,dx - 2\sum_j\int_0^1 v^2\langle u_j^h,\mathcal{Q}\sigma_j\rangle\,\mathrm{d}x\,dt + \sum_{i,j}\int_0^1 v^2\langle u_i^h,u_j^h\rangle\langle\mathcal{Q}\sigma_i,\sigma_j\rangle\,\mathrm{d}x\,dt\\ \leq \eta_0\|v\|^2\,dt + c\varepsilon^{-1/2}\varepsilon^{1/2}\eta_0\|v\|^2\,dt + c\varepsilon^{-1}\varepsilon^{1/2}\eta_0\varepsilon^{1/2}\|v\|^2\,dt \;=\; \mathcal{O}(\eta_0)\|v\|^2\,dt,\end{aligned}$$

where we utilized the estimates of Proposition 5.7 for the derivatives of u^h together with the bound on the diffusion σ by Lemma 5.21. □

As a next step, we study the critical term $\langle v^3, dv\rangle$. Note that we focus for simplicity only on the toy problem $f(u) = u^3 - u$. Later in Section 5.6, we extend our analysis to general nonlinearities given by polynomials of odd degree. Expanding dv yields

$$\begin{aligned}\langle v^3,dv\rangle \;&=\; \langle\mathcal{L}(u^h),v^3\rangle\,dt + \varepsilon^2\int_0^1 v^3 v_{xx}\,\mathrm{d}x\,dt + \int_0^1\Big(1-3(u^h)^2\Big)v^4\,\mathrm{d}x\,dt\\ &\quad -\int_0^1 3u^h v^5\,\mathrm{d}x\,dt - \int_0^1 v^6\,\mathrm{d}x\,dt + \langle v^3,dW\rangle - \langle v^3,du^h\rangle\\ &=\; -3\varepsilon^2\int_0^1 v^2v_x^2\,\mathrm{d}x\,dt - \|v\|_{L^6}^6\,dt + \langle\mathcal{L}(u^h),v^3\rangle\,dt + \int_0^1\Big(1-3(u^h)^2\Big)v^4\,\mathrm{d}x\,dt\\ &\quad -\int_0^1 3u^hv^5\,\mathrm{d}x\,dt + \langle v^3,dW\rangle - \langle v^3,du^h\rangle.\end{aligned}$$

We see that the good (negative) terms for our analysis are given by $-\|v\|_{L^6}^6$ and, due to integration by parts,

$$\varepsilon^2\int_0^1 v^3v_{xx}\,\mathrm{d}x \;=\; -3\varepsilon^2\int_0^1 v^2v_x^2\,\mathrm{d}x \;=\; -\frac{3}{4}\varepsilon^2\int_0^1((v^2)_x)^2\,\mathrm{d}x \;=\; -\frac{3}{4}\varepsilon^2\|(v^2)_x\|^2.$$

Our strategy is to absorb as much as possible of the remaining terms into these negative ones, while also using that we can control the L^2-norm by the preceding stability result. We begin with analyzing the dominant term. Since u^h is uniformly bounded, we obtain by interpolating the L^4-norm between the good terms

$$\begin{aligned}\int_0^1\Big(1-3(u^h)^2\Big)v^4\,\mathrm{d}x \;&\leq\; C\|v\|_{L^4}^4 \;\leq\; C\|v^2\|_{L^\infty}\int_0^1 v^2\,\mathrm{d}x \overset{\text{Agmon}}{\leq} C\|v\|^2\|v^2\|_{H^1}^{1/2}\|v^2\|^{1/2}\\ &\overset{\text{Young}}{\leq} \frac{1}{8}\varepsilon^2\|v^2\|_{H^1}^2 + c\varepsilon^{-2/3}\|v\|^{8/3}\|v\|_{L^4}^{4/3}\\ &\overset{\text{Hölder}}{\leq} \frac{1}{8}\varepsilon^2\|v^2\|_{H^1}^2 + c\varepsilon^{-2/3}\|v\|^3\|v\|_{L^6}\\ &\overset{\text{Young}}{\leq} \frac{1}{8}\varepsilon^2\|v^2\|_{H^1}^2 + \frac{1}{4}\|v\|_{L^6}^6 + c\varepsilon^{-4/5}\|v\|^{18/5}.\end{aligned}\tag{5.32}$$

Similarly, we estimate the L^5-term

$$\begin{aligned}\int_0^1 3u^hv^5 \;&\leq\; 3\|v\|_{L^5}^5 \;\leq\; c\|v\|_{L^3}^3\|v^2\|_{L^\infty} \overset{\text{Agmon}}{\leq} c\|v\|_{L^3}^3\|v^2\|_{H^1}^{1/2}\|v^2\|^{1/2}\\ &\overset{\text{Young}}{\leq} \frac{1}{8}\varepsilon^2\|v^2\|_{H^1}^2 + c\varepsilon^{-2/3}\|v\|_{L^3}^4\|v^2\|^{2/3}\\ &= \frac{1}{8}\varepsilon^2\|v^2\|_{H^1}^2 + c\varepsilon^{-2/3}\|v\|_{L^3}^4\|v\|_{L^4}^{4/3}\\ &\overset{\text{Hölder}}{\leq} \frac{1}{8}\varepsilon^2\|v^2\|_{H^1}^2 + c\varepsilon^{-2/3}\|v\|^{7/3}\|v\|_{L^6}^3\\ &\overset{\text{Young}}{\leq} \frac{1}{8}\varepsilon^2\|v^2\|_{H^1}^2 + \frac{1}{4}\|v\|_{L^6}^6 + c\varepsilon^{-4/3}\|v\|^{14/3}.\end{aligned}\tag{5.33}$$

Combining the previous estimates we derived so far

$$\begin{aligned}\langle v^3, dv\rangle \;\leq\; & -\frac{1}{2}\varepsilon^2\|v^2\|_{H^1}^2\,dt - \frac{1}{2}\|v\|_{L^6}^6\,dt + \left[c\varepsilon^{-4/5}\|v\|^{18/5} + c\varepsilon^{-16/7}\|v\|^{66/7}\right]dt \\ & + \langle v^3, dW\rangle - \langle v^3, du^h\rangle.\end{aligned}$$

Note that we used $\mathcal{L}(u^h) = \mathcal{O}(\exp)$ and thus $\langle \mathcal{L}(u^h), v^3\rangle \leq \mathcal{O}(\exp) + \mathcal{O}(\exp)\|v\|_{L^6}^6$ by Hölder's inequality. Finally, we estimate $\langle v^3, du^h\rangle$ given by

$$\langle v^3, du^h\rangle \;=\; \sum_j \langle v^3, u_j^h\rangle\,dh_j + \frac{1}{2}\sum_{i,j}\langle v^3, u_{ij}^h\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle\,dt.$$

For the second summand we have

$$\begin{aligned}\langle v^3, u_{ij}^h\rangle\langle \mathcal{Q}\sigma_i, \sigma_j\rangle \;&\leq\; \|v^2\|_{L^\infty}\|v\|\varepsilon^{-3/2}\eta_0\varepsilon^{1/2}\varepsilon^{1/2} \\ &\overset{\text{Agmon}}{\leq} \|v^2\|_{H^1}^{1/2}\|v^2\|^{1/2}\|v\|\varepsilon^{-1/2}\eta_0 \\ &\overset{\text{Young}}{\leq} \frac{1}{8}\varepsilon^2\|v^2\|_{H^1}^2 + c\varepsilon^{-2/3}\eta_0^{4/3}\|v\|^{4/3}\|v\|_{L^4}^{4/3} \\ &\overset{\text{Hölder}}{\leq} \frac{1}{8}\varepsilon^2\|v^2\|_{H^1}^2 + c\varepsilon^{-2/3}\eta_0^{4/3}\|v\|^{5/3}\|v\|_{L^6} \\ &\overset{\text{Young}}{\leq} \frac{1}{4}\varepsilon^2\|v^2\|_{H^1}^2 + \frac{1}{8}\|v\|_{L^6}^6 + c\varepsilon^{-4/5}\eta_0^{8/5}\|v\|^2.\end{aligned} \tag{5.34}$$

In addition to the specified inequalities, we used that $\|u_{ij}^h\| = \mathcal{O}(\varepsilon^{-3/2})$ by Proposition 5.7 and $\|\sigma\| = \mathcal{O}(\varepsilon^{1/2})$ by Lemma 5.21.

We conclude by analyzing the term involving the stochastic differential dh. Recall that by (5.18) $dh_j = b_j(h,v)\,dt + \langle \sigma_j(h,v), dW\rangle$, where b and σ are given by (5.20) and (5.19), respectively. The diffusion term of $\langle v^3, u_j^h\rangle\,dh_j$ can be estimated as follows:

$$\begin{aligned}\langle v^3, u_j^h\rangle\langle \sigma_j, dW\rangle \;&=\; \langle \mathcal{O}(\|u_j^h\|_{L^\infty}\|\sigma_j\|\|v\|_{L^3}^3), dW\rangle \\ &=\; \langle \mathcal{O}(\varepsilon^{-1/2}\|v\|_{L^3}^3), dW\rangle \\ &=\; \langle \mathcal{O}(\varepsilon^{-1/2}\|v\|\|v\|_{L^4}^2), dW\rangle.\end{aligned}$$

Estimating the drift of $\langle v^3, u_j^h\rangle\,dh_j$ is trickier, as we have to bound b as well. We have seen in Lemmata 5.21 and 5.22 that we can bound b up to a stopping time. As long as $\|v(t)\| < \varepsilon^{1/2+m}$ for some $m > 0$, we obtain

$$|b_j| \;=\; \mathcal{O}(\eta_0) + \mathcal{O}(\varepsilon)\,|\langle \mathcal{N}^h(v), u_j^h\rangle|,$$

where the inner product involving the nonlinearity can be estimated by

$$\begin{aligned}\langle \mathcal{N}^h(v), u_j^h\rangle \;&=\; \int_0^1 \left(3u^h v^2 - v^3\right)u_i^h\,\mathrm{d}x \\ &\leq\; \|u^h\|_{L^\infty}\|u_j^h\|_{L^\infty}\|v\|^2 + \|u_j^h\|_{L^\infty}\|v\|_{L^3}^3 \\ &\leq\; c\varepsilon^{-1}\|v\|^2 + c\varepsilon^{-1}\|v\|\|v\|_{L^4}^2.\end{aligned}$$

This yields $|b_j| \leq \mathcal{O}(\eta_0) + c\|v\|^2 + c\|v\|\|v\|_{L^4}^2$, and we obtain

$$\begin{aligned}|\langle v^3, u_j^h\rangle||b_j| \;&\leq\; c\varepsilon^{-1}\eta_0\|v\|_{L^3}^3 + c\varepsilon^{-1/2}\|v\|^2\|v\|_{L^6}^3 + c\varepsilon^{-1/2}\|v\|\|v\|_{L^4}^2\|v\|_{L^6}^3 \\ &\overset{\text{Hölder}}{\leq} c\varepsilon^{-1}\eta_0\|v\|^{3/2}\|v\|_{L^6}^{3/2} + c\varepsilon^{-1/2}\|v\|^2\|v\|_{L^6}^3 + c\varepsilon^{-1/2}\|v\|^{3/2}\|v\|_{L^6}^{9/2} \\ &\overset{\text{Young}}{\leq} \frac{1}{8}\|v\|_{L^6}^6 + c\varepsilon^{-4/3}\eta_0^{4/3}\|v\|^2 + c\varepsilon^{-1}\|v\|^4 + c\varepsilon^{-2}\|v\|^6.\end{aligned} \tag{5.35}$$

Finally, we estimated every term of $d\|v\|_{L^4}^4$. Thus far, our estimates depend on the L^2-norm of v. Under the assumptions of Theorem 5.36, i.e., a small noise strength η_0 and a suitable initial condition $v(0)$, we can bound $\|v\|_{L^2}$ by an ε-dependent constant for long time scales. In more detail, we obtain the following result.

Theorem 5.41. *As long as $\|v\| \leq \varepsilon^{1/2+m}$ and $\eta_0 \leq \varepsilon^{1+2m}$ for some $m > 0$, we have*

$$\frac{1}{4} d\|v\|_{L^4}^4 \;\leq\; \left[-\frac{1}{2}\|v\|_{L^4}^4 + c\varepsilon^{2m+1}\right] dt + \langle \mathcal{O}(\varepsilon^m \|v\|_{L^4}^2 + \|v\|_{L^4}^3), dW\rangle.$$

Proof. For the proof, one essentially follows the preceding estimates and uses that by Hölder's inequality

$$\|v\|_{L^4}^4 \;\leq\; \|v\|\|v\|_{L^6}^3 \;\leq\; \frac{1}{2}\|v\|^2 + \frac{1}{2}\|v\|_{L^6}^6 \;\leq\; \frac{1}{2}\|v\|_{L^6}^6 + c\varepsilon^{1+2m}.$$

Moreover, let us denote by $K_\varepsilon(\|v\|)$ all terms appearing in the previous estimates of the stochastic differential $d\|v\|_{L^4}^4$, which only depend on $\|v\|$. By Lemma 5.40 and the estimates (5.32), (5.33), (5.34), and (5.35), these terms are given by

$$\begin{aligned} K_\varepsilon(\|v\|) \;&=\; C\left(\varepsilon^{-4/5}\|v\|^{8/5} + \varepsilon^{-4/3}\|v\|^{8/3} + \varepsilon^{-1}\|v\|^2 + \varepsilon^{-2}\|v\|^4\right)\|v\|^2 \\ &\quad + \left(\eta_0 + \varepsilon^{-4/5}\eta_0^2 + \varepsilon^{-4/3}\eta_0^{4/3}\right)\|v\|^2. \end{aligned}$$

By the assumptions on η_0 and $\|v\|$, one easily computes that $K_\varepsilon(\|v\|) \leq \varepsilon^{8m/5}\|v\|^2 \leq \varepsilon^{1+2m}$. Note that the dominating term in $K_\varepsilon(\|v\|)$ is arising from estimate (5.32). $\square$

With this inequality at hand, we can apply the main stability theorem 2.14 of Chapter 2. Bear in mind that in the derivation of Theorem 5.41 we presented only one technique and thus, we cannot guarantee the optimality of the radii.

Theorem 5.42 (L^4-Stability for (AC))**.**
For $m > 0$ and small $\kappa > 0$, consider the stopping time

$$\tau^* \;=\; \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0] \;:\; \|v(t)\| \;>\; \varepsilon^{1/2+m} \quad \text{or} \quad \|v(t)\|_{L^4} \;>\; \varepsilon^{1/4+m/2-\kappa}\right\},$$

where $T_\varepsilon = \varepsilon^{-M}$ for any fixed large $M > 0$ and τ_0 denotes the first exit time from Ω_ρ. Also, assume that for some $\nu \in (0, 1)$

$$\|v(0)\| \;\leq\; \nu\varepsilon^{1/2+m} \quad \text{and} \quad \|v(0)\|_{L^4} \;\leq\; \nu\varepsilon^{1/4+m/2-\kappa}$$

and that for the squared noise strength

$$\eta_0 \;\leq\; \varepsilon^{1+2m}.$$

Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as ε tends to zero.

Proof. The estimate of the diffusion term in Theorem 5.41 does not quite fit in the setting of the general stability result of Theorem 2.14. For this term, we obtain up to the stopping time τ^*

$$\varepsilon^m\|v\|_{L^4}^2 + \|v\|_{L^4}^3 \;\leq\; c\varepsilon^{\min\{m\,,\,1/4+m/2-\kappa\}}\|v\|_{L^4}^2.$$

By setting $x(t) = \|v(t)\|_{L^4}^2$ and utilizing this estimate together with Theorem 5.41, one then obtains

$$dx(t)^2 \;\leq\; \left[K_\varepsilon(\eta_0) - a_\varepsilon x(t)^2\right] dt + \langle \mathcal{O}(c_\varepsilon x(t)), dW\rangle,$$

where $a_\varepsilon = \frac{1}{2}$, $K_\varepsilon(\eta_0) = \mathcal{O}(\varepsilon^{2m+1})$, and $c_\varepsilon = \mathcal{O}(\varepsilon^{\min\{m\,,1/4+m/2-\kappa\}})$.
With this stochastic differential inequality, we are back in the setting of Theorem 2.14. Note that we now have to adept the radius due to the substitution. For the result with respect to x we have to consider R_ε^2, where $R_\varepsilon = \varepsilon^{1/4+m/2-\kappa}$ is the given L^4-radius in the definition of τ^*. With that, one readily verifies that

$$\frac{K_\varepsilon(\eta_0) + c_\varepsilon^2 \eta_1}{a_\varepsilon R_\varepsilon^4} = \mathcal{O}(\varepsilon^{4\kappa}).$$

So far, this shows that $\mathbb{P}(\|v(\tau^*)\|_{L^4} > \varepsilon^{1/4+m/2-\kappa})$ is smaller than any power of ε. By the L^2-result of Theorem 5.36 and the basic inequality

$$\mathbb{P}\left(\tau_\varepsilon < T_\varepsilon \wedge \tau_0\right) \le \mathbb{P}\left(\|v(\tau_\varepsilon)\| > \varepsilon^{1/2+m}\right) + \mathbb{P}\left(\|v(\tau^*)\|_{L^4} > \varepsilon^{1/4+m/2-\kappa}\right),$$

the proof is complete. □

5.5.2 L^4-Stability for (mAC)

We conclude this section with establishing stochastic L^4-stability in the mass conserving case (mAC). As usual, we start with an analogy to the stochastic differential inequality in Theorem 5.41. Due to the different radius R_ε and the noise strength η_0 of the L^2-stability result of Theorem 5.39, we have to adept the ε-dependent constants.

Corollary 5.43. *As long as $\|v\| \le \varepsilon^{3/2+m}$ and $\eta_0 \le \varepsilon^{4+2m}$ for some $m > 0$, we have*

$$\frac{1}{4} d\|v\|_{L^4}^4 \le \left[-\frac{1}{2}\|v\|_{L^4}^4 + c\varepsilon^{3+2m}\right] dt + \langle \mathcal{O}(\varepsilon^{1+m}\|v\|_{L^4}^2 + \|v\|_{L^4}^3), dW\rangle.$$

Proof. The derivation leading to Theorem 5.41 does not change in the mass conserving case, since all quantities can be estimated in the same way. One easily verifies that the dominating term in $K_\varepsilon(\|v\|)$ remains the same and thus

$$K_\varepsilon(\|v\|) < c\varepsilon^{-4/5}\|v\|^{18/5} < c\varepsilon^{8/5+8m/5}\|v\|^2 < \|v\|^2.$$ □

With the estimate from Corollary 5.43, we can again rely on the general theorem 2.14 to derive stochastic stability.

Theorem 5.44 (L^4-Stability for (mAC))**.**
For $m > 0$ and small $\kappa > 0$, consider the stopping time

$$\tau^* = \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0] \,:\, \|v(t)\| > \varepsilon^{3/2+m} \;\; or \;\; \|v(t)\|_{L^4} > \varepsilon^{3/4+m/2-\kappa}\right\},$$

where $T_\varepsilon = \varepsilon^{-M}$ for fixed large $M > 0$ and τ_0 denotes the first exit time from the set of admissible positions $\mathcal{A}_\rho$. Also, assume that for some $\nu \in (0,1)$

$$\|v(0)\| \le \nu\varepsilon^{3/2+m} \quad and \quad \|v(0)\|_{L^4} \le \nu\varepsilon^{3/4+m/2-\kappa}$$

and that the squared noise strength satisfies $\eta_0 \le \varepsilon^{4+2m}$.
Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as ε tends to zero.

Proof. Here, we can essentially follow the proof of Theorem 5.42. Using the adapted versions of the constants, one then also obtains $\frac{K_\varepsilon(\eta_0)+c_\varepsilon^2\eta_1}{a_\varepsilon R_\varepsilon^4} = \mathcal{O}(\varepsilon^{4\kappa})$. The statement follows now from the corresponding L^2-result in Theorem 5.39. □

5.6 Extension to general nonlinearites

In the preceding stability analysis, we assumed that f is the derivative of the standard quartic double-well potential $F(u) = \frac{1}{4}(u^2-1)^2$. As the construction of the slow manifold and the spectral analysis apply for a broader class of nonlinearities, we extend the results. In this section, we assume that the nonlinearity $f = F'$ is given by a polynomial of odd degree with positive leading coefficient, that is,

$$f(x) \;=\; a_{2p-1}x^{2p-1} + \sum_{k=1}^{2p-2} a_k x^k \qquad \text{for } p \in \mathbb{N},\; a_{2p-1} > 0, \text{ and } a_1,\dots,a_{2p-2} \in \mathbb{R}. \tag{5.36}$$

Crucial for the analysis and well-definedness of the stochastic ODE governing the interface motion are bounds on the nonlinearity

$$\mathcal{N}^h(v) \;=\; f(u^h) - f(u^h+v) + f'(u^h)v \;=\; -a_{2p-1}v^{2p-1} + \sum_{k=2}^{2p-2} c_k v^k. \tag{5.37}$$

Obviously, we can find a positive constant C such that pointwise

$$|\mathcal{N}^h(v)| \;\le\; C\left(|v|^{2p} + |v|^2\right).$$

Hence, in order to control $\mathcal{N}^h(v)$ in L^1, it is sufficient to control the L^{2p}-norm of v. Note that for our argument it is important that we have an even power (see Subsections 5.5.1 and 5.5.2). Let us first show that the L^2-stability result of Theorem 5.39 still holds true for this class of nonlinearities. Essential for our technique of proof was the estimate of Theorem 5.33, namely

$$\langle \mathcal{L}^h v + \mathcal{N}^h(v), v\rangle \;\le\; -\frac{1}{2}\lambda_0\|v\|^2,$$

for some $\lambda_0 > 0$ and v orthogonal to the tangent space of $\mathcal{M}$ at u^h. In the proof of this bound, we needed the estimate $\langle \mathcal{N}^h(v), v\rangle \le C\|v\|^3_{L^3}$. Note that this is the only term, which is influenced by the change of the nonlinearity. Let us briefly show that this still holds true in the general case and thus the L^2-result of Theorem 5.39 remains valid.

Lemma 5.45. *Let f be the odd polynomial given by (5.36) and $\mathcal{N}^h(v)$ the nonlinearity defined by (5.37). Then, we have*

$$\langle \mathcal{N}^h(v), v\rangle \;\le\; C\|v\|^3_{L^3}.$$

Proof. By (5.37) we find $c_1,\dots,c_{2p-3} \in \mathbb{R}$ such that

$$\mathcal{N}^h(v) \;=\; -a_{2p-1}v^{2p-1} + \left(\sum_{k=1}^{2p-3} c_k v^k\right) v.$$

Let $C > 0$ be such that

$$\Big|\sum_{k=1}^{2p-3} c_k v^k\Big| \;\le\; \frac{1}{2}a_{2p-1}v^{2p-2} + C|v|.$$

Then, we obtain

$$\langle \mathcal{N}^h(v), v\rangle \;=\; \int_0^1 -a_{2p-1}v^{2p} + \left(\sum_{k=1}^{2p-3} c_k v^k\right) v^2\,\mathrm{d}x \;\le\; \int_0^1 -\frac{1}{2}a_{2p-1}v^{2p} + C|v|^3\,\mathrm{d}x \;\le\; C\|v\|^3_{L^3}. \qquad \square$$

Therewith, we are in the setting of the preceding section and can follow the proof that first led to the inequality for $d\|v\|^2$ of Corollary 5.35, and then to the L^2-stability result of Theorem 5.39. Hence, the following result is furnished in the general case.

Corollary 5.46 (L^2-Stability for general nonlinearities)**.**
For $m > 0$ define the stopping time

$$\tau^* = \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0] : \|v(t)\| > \varepsilon^{1/2+m}\right\},$$

where $T_\varepsilon = \varepsilon^{-M}$ for fixed large $M > 0$ and τ_0 denotes the first exit time from Ω_ρ defined in (5.31). Also, assume that for some $\nu \in (0,1)$

$$\|v(0)\| \le \nu\varepsilon^{1/2+m} \quad \text{and} \quad \eta_0 \le \varepsilon^{1+2m}.$$

Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as ε tends to zero.

Similarly to the analysis of the quartic potential, we use the long-time stability in L^2 to extend the stability result to L^{2p}. Recall that by Theorem 5.39 our optimal radius with respect to the L^2-norm is given by $R_\varepsilon = \varepsilon^{1/2+m}$ for a squared noise strength of order $\mathcal{O}(\varepsilon^{1+2m})$. We have seen in Theorem 5.42 that the radius in the L^4-setting scales like $R_\varepsilon^{1/2}$, while we can rely on the same noise strength η_0. In the general case, we expect that the optimal L^{2p}-radius should behave like $R_\varepsilon^{1/p}$. Essential for deriving stochastic stability in L^{2p} is the following theorem. Here, we will assume $p \ge 3$, since we already proved the result in L^2 and L^4.

Theorem 5.47. *Let $p \ge 3$. Furthermore, assume that $\eta_0 \le \varepsilon^{1+2m}$ and $\|v\|_{L^2} \le \varepsilon^{1/2+m}$ for some $m > 0$. Then, we obtain for a constant $c_p > 0$ depending only on p*

$$d\|v\|_{L^{2p}}^{2p} \le c_p\left[-\frac{1}{2}\|v\|_{L^{2p}}^{2p} - \frac{1}{2}\|v\|_{L^{4p-2}}^{4p-2} - \varepsilon^2\|(v^p)_x\|^2 + \|v\|^2\right] dt + \langle\psi, dW\rangle,$$

where

$$\|\psi\|_{L^2}^2 \le c\left(\varepsilon\|v^p\|_{H^1}^2 + \varepsilon^{-1}\|v\|_{L^{4p-2}}^{4p-2} + \varepsilon^{-1}\|v\|^2\right)\|v\|_{L^{2p}}^{2p}.$$

Proof. Here, we follow the method that led to Theorem 5.41 very closely. By Itô formula we obtain

$$\frac{1}{2p}d\|v\|_{L^{2p}}^{2p} = \langle v^{2p-1}, dv\rangle + (2p-1)\int_0^1 v^{2p-2}(dv)^2\,\mathrm{d}x.$$

Similarly to Lemma 5.40, we see that

$$\int_0^1 v^{2p-2}(dv)^2\,\mathrm{d}x \le C\eta_0\|v\|_{L^{2p-2}}^{2p-2} \le C\eta_0\|v\|^{\frac{p}{p-1}}\|v\|_{L^{4p-2}}^{\frac{(2p-1)(p-2)}{p-1}} \le c\|v\|_{L^{4p-2}}^{4p-2} + c\eta_0^{2-2/p}\|v\|^2,$$

where we used Hölder's inequality to interpolate the L^{2p-2}-norm. Also note that by assumption $\eta_0^{2-2/p} < 1$. We continue with the critical terms. By expansion of dv we have

$$\begin{aligned}\langle v^{2p-1}, dv\rangle &= \varepsilon^2\int_0^1 v^{2p-1}v_{xx}\,\mathrm{d}x\,dt + \langle v^{2p-1}, f(u^h) - f(u^h+v)\rangle\,dt \\ &\quad - \langle v^{2p-1}, \mathcal{L}(u^h)\rangle\,dt - \langle v^{2p-1}, du^h\rangle + \langle v^{2p-1}, dW\rangle.\end{aligned}$$

Integration by parts yields for the first term

$$\varepsilon^2\int_0^1 v^{2p-1}v_{xx} = -(2p-1)\varepsilon^2\int_0^1 v^{2p-2}v_x^2 = -\frac{2p-1}{p^2}\varepsilon^2\int_0^1((v^p)_x)^2 = -\frac{2p-1}{p^2}\varepsilon^2\|(v^p)_x\|^2,$$

which is a good term for our analysis. We choose $c > 0$ such that

$$f(u^h) - f(u^h+v) \le -a_{2p-1}v^{2p-1} + cv.$$

We obtain

$$\langle v^{2p-1}, f(u^h) - f(u^h+v)\rangle \le -a_{2p-1}\|v\|_{L^{4p-2}}^{4p-2} + c\|v\|_{L^{2p}}^{2p}.$$

Hence, this estimate furnished another good negative L^{4p-2}-term. We estimate the L^{2p}-term further and absorb as much as possible into the negative terms. By Agmon's and Young's inequality we derive

$$\|v\|_{L^{2p}}^{2p} \le \int_0^1 \|v^p\|_\infty |v|^p \,\mathrm{d}x \le \|v^p\|_{H^1}^{1/2} \|v\|_{L^{2p}}^{p/2} \|v\|_{L^p}^p \le c\varepsilon^2 \|v^p\|_{H^1}^2 + c\varepsilon^{-2/3} \|v\|_{L^{2p}}^{2p/3} \|v\|_{L^p}^{4p/3}.$$

We interpolate the Lebesgue spaces between L^2 and L^{4p-2}. With Hölder's inequality we obtain

$$\|v\|_{L^{2p}}^{2p/3} \le \|v\|_{L^2}^{1/3} \|v\|_{L^{4p-2}}^{(2p-1)/3} \quad \text{and} \quad \|v\|_{L^p}^{4p/3} \le \|v\|_{L^2}^{\frac{2}{3}\frac{3p-2}{p-1}} \|v\|_{L^{4p-2}}^{\frac{2}{3}\frac{(p-2)(2p-1)}{p-1}}.$$

Combining the interpolation estimates and using Young's inequality yields

$$\begin{aligned} c\varepsilon^{-2/3} \|v\|_{L^{2p}}^{2p/3} \|v\|_{L^p}^{4p/3} &\le c\varepsilon^{-2/3} \|v\|_{L^2}^{\frac{7p-5}{3p-3}} \|v\|_{L^{4p-2}}^{\frac{(2p-1)(3p-5)}{3p-3}} \\ &\le c\|v\|_{L^{4p-2}}^{4p-2} + c\varepsilon^{-4\frac{p-1}{3p-1}} \|v\|_{L^2}^{2\frac{7p-5}{3p-1}} = c\|v\|_{L^{4p-2}}^{4p-2} + c\varepsilon^{-4\frac{p-1}{3p-1}} \|v\|_{L^2}^{8\frac{p-1}{3p-1}} \|v\|_{L^2}^2. \end{aligned}$$

We observe that by the assumption on $\|v\|_{L^2}$ we always have $\varepsilon^{-4\frac{p-1}{3p-1}} \|v\|_{L^2}^{8\frac{p-1}{3p-1}} < \varepsilon^{8m\frac{p-1}{3p-1}} < 1$. So far, we have shown that

$$\varepsilon^2 \int_0^1 v^{2p-1} v_{xx} \,\mathrm{d}x + \langle v^{2p-1}, f(u^h) - f(u^h + v)\rangle \le -C\varepsilon^2 \|(v^p)_x\|^2 - C\|v\|_{L^{4p-2}}^{4p-2} + C\|v\|^2.$$

By Proposition 5.7 we see that $\mathcal{L}(u^h) = \mathcal{O}(\exp)$ and thus

$$\langle v^{2p-1}, \mathcal{L}(u^h)\rangle = \mathcal{O}(\exp) \|v\|_{L^{4p-2}}^{4p-2} + \mathcal{O}(\exp).$$

Next, we control the term $\langle v^{2p-1}, du^h\rangle$ given by

$$\sum_j \langle v^{2p-1}, u_j^h\rangle \, dh_j + \frac{1}{2} \sum_{i,j} \langle v^{2p-1}, u_{ij}^h\rangle \langle \mathcal{Q}\sigma_i, \sigma_j\rangle \, dt.$$

For the second term, we obtain

$$\begin{aligned} |\langle v^{2p-1}, u_{ij}^h\rangle \langle \mathcal{Q}\sigma_i, \sigma_j\rangle| &\le c\varepsilon^{-1/2} \eta_0 \|v\|_{L^{4p-2}}^{2p-1} \le c\varepsilon^{-1/2} \eta_0 \|v^p\|_\infty \|v\|_{L^{2p-2}}^{p-1} \\ &\le c\varepsilon^{-1/2} \eta_0 \|v^p\|_{H^1}^{1/2} \|v\|_{L^{2p}}^{p/2} \|v\|_{L^{2p-2}}^{p-1} \\ &\le c\varepsilon^2 \|v^p\|_{H^1}^2 + c\varepsilon^{-2/3} \eta_0^{4/3} \|v\|_{L^{2p}}^{2p/3} \|v\|_{L^{2p-2}}^{4\frac{p-1}{3}} \\ &\le c\varepsilon^2 \|v^p\|_{H^1}^2 + c\varepsilon^{-2/3} \eta_0^{4/3} \|v\|_{L^2}^{\frac{3p-1}{3p-3}} \|v\|_{L^{4p-2}}^{\frac{(2p-1)(3p-5)}{3p-3}} \\ &\le c\varepsilon^2 \|v^p\|_{H^1}^2 + c\|v\|_{L^{4p-2}}^{4p-2} + c\varepsilon^{-4\frac{p-1}{3p-1}} \eta_0^{8\frac{p-1}{3p-1}} \|v\|^2. \end{aligned} \tag{5.38}$$

Here, the prefactor of the L^2-norm is smaller than 1 as $\eta_0 < \varepsilon^{1+2m}$.
We have established in Lemmata 5.21 and 5.22 that, as long as $\|v\| < \varepsilon^{1/2+m}$, the drift term b of dh is bounded by

$$|b| = \mathcal{O}(\eta_0) + \mathcal{O}(\varepsilon) |\langle \mathcal{N}^h(v), u_j^h\rangle|.$$

Since $\mathcal{N}^h(v)$ is a polynomial in v of degree $2p-1$ with uniformly bounded coefficients, we find a constant $C > 0$ such that $|\mathcal{N}^h(v)| \le C(|v|^{2p-1} + 1)$ and therefore

$$|b| \le C\eta_0 + C\varepsilon^{1/2} \|v\|_{L^{4p-2}}^{2p-1} + C\varepsilon^{1/2}.$$

This leads to the following estimate

$$|\langle v^{2p-1}, u_j^h\rangle| |b_j| \le C(\eta_0 \varepsilon^{-1/2} + 1) \|v\|_{L^{4p-2}}^{2p-1} + C\varepsilon^{-1/2} \|v\|_{L^{2p-1}}^{2p-1} \|v\|_{L^{4p-2}}^{2p-1}.$$

The first summand on the right-hand side can be treated identical to (5.38).

For the second one, we obtain via Hölder's and Young's inequality

$$\begin{aligned} C\varepsilon^{-1/2}\|v\|_{L^{2p-1}}^{2p-1}\|v\|_{L^{4p-2}}^{2p-1} &\leq C\varepsilon^{-1/2}\|v\|^{\frac{2p-1}{2p-2}}\|v\|_{L^{4p-2}}^{\frac{(2p-1)(4p-5)}{2p-2}} \\ &\leq c\|v\|_{L^{4p-2}}^{4p-2} + c\varepsilon^{-2(p-1)}\|v\|^{2(2p-1)} = c\|v\|_{L^{4p-2}}^{4p-2} + c\varepsilon^{-2(p-1)}\|v\|^{4(p-1)}\|v\|^2. \end{aligned}$$

And once again, by assumption the prefactor of the L^2-term is smaller than 1.
Let us summarize all the estimates we achieved so far. For a constant C_p depending only on p, we proved that

$$\begin{aligned} d\|v\|_{L^{2p}}^{2p} &\leq C_p\left[-\|v\|_{L^{4p-2}}^{4p-2} - \varepsilon^2\|(v^p)_x\|^2 + \|v\|_{L^2}^2\right] dt + \langle v^{2p-1} + \langle v^{2p-1}, u_j^h\rangle\sigma, dW\rangle \\ &\leq C_p\left[-\tfrac{1}{2}\|v\|_{L^{2p}}^{2p} - \tfrac{1}{2}\|v\|_{L^{4p-2}}^{4p-2} - \varepsilon^2\|(v^p)_x\|^2 + \|v\|_{L^2}^2\right] dt + \langle v^{2p-1} + \langle v^{2p-1}, u_j^h\rangle\sigma, dW\rangle. \end{aligned}$$

In the last step, we utilized that by Hölder's and Young's inequality

$$\|v\|_{L^{2p}}^{2p} \leq \|v\|_{L^2}\|v\|_{L^{4p-2}}^{2p-1} \leq \tfrac{1}{2}\|v\|_{L^2}^2 + \tfrac{1}{2}\|v\|_{L^{4p-2}}^{4p-2}.$$

It remains to control the diffusion

$$\psi_j = v^{2p-1} + \langle v^{2p-1}, u_j^h\rangle\sigma_j.$$

We have

$$\begin{aligned} \|\psi_j\|_{L^2}^2 &= \|v\|_{L^{4p-2}}^{4p-2} + 2\langle v^{2p-1}, u_j^h\rangle\langle v^{2p-1}, \sigma_j\rangle + \langle v^{2p-1}, u_j^h\rangle^2\|\sigma_j\|_{L^2}^2 \\ &\leq C\|v\|_{L^{4p-2}}^{4p-2} \leq C\|v^p\|_\infty^2\|v\|_{L^{2p-2}}^{2p-2} \\ &\leq C\|v^p\|_{H^1}\|v\|_{L^{2p}}^p\|v\|_{L^{2p-2}}^{2p-2} \\ &\leq C\left(\|v^p\|_{H^1}\|v\|_{L^{2p}}^{\frac{p(p-3)}{p-1}}\|v\|^{\frac{2}{p-1}}\right)\|v\|_{L^{2p}}^{2p} \\ &\leq C\left(\|v^p\|_{H^1}\|v\|_{L^{4p-2}}^{\frac{(2p-1)(p-3)}{2(p-1)}}\|v\|^{\frac{p+1}{2(p-1)}}\right)\|v\|_{L^{2p}}^{2p} \\ &\leq \left(c\varepsilon\|v^p\|_{H^1}^2 + \varepsilon^{-1}\|v\|_{L^{4p-2}}^{\frac{(2p-1)(p-3)}{p-1}}\|v\|^{\frac{p+1}{p-1}}\right)\|v\|_{L^{2p}}^{2p} \\ &\leq \left(c\varepsilon\|v^p\|_{H^1}^2 + \varepsilon^{-1}\|v\|_{L^{4p-2}}^{4p-2} + \varepsilon^{-1}\|v\|^2\right)\|v\|_{L^{2p}}^{2p}. \end{aligned}$$

This concludes the proof. □

With help of Theorem 5.47, we can prove the long-time stability for general nonlinearities. The following theorem verifies that the maximal radius in L^{2p} is indeed given by $R_\varepsilon^{1/p}$, where R_ε denotes the L^2-radius. This leads to a generalization of the result we obtained in Theorem 5.42.

Theorem 5.48 (L^{2p}-stability for general nonlinearities)**.**
For $m > 0$, small $\kappa > 0$, and $p \geq 3$, consider the stopping time

$$\tau^* = \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0] : \|v(t)\| > \varepsilon^{1/2+m} \ \text{ or } \ \|v(t)\|_{L^{2p}} > \varepsilon^{1/2p+m/p-\kappa}\right\}, \tag{5.39}$$

where $T_\varepsilon = \varepsilon^{-M}$ for fixed large $M > 0$. Moreover, assume that for some $\nu \in (0,1)$

$$\|v(0)\| \leq \nu\varepsilon^{1/2+m} \quad \text{and} \quad \|v(0)\|_{L^{2p}} \leq \nu\varepsilon^{1/2p+m/p-\kappa},$$

and that for the squared noise strength $\eta_0 \leq \varepsilon^{1+2m}$.
Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as ε tends to zero.

Proof. The estimate of the diffusion term ψ in Theorem 5.47 does not quite fit in the setting of Theorem 2.14. In order to apply this theorem, we have to bound positive powers of $\|v\|_{L^{2p}}^{2p}$. In the sequel, we will use C for a positive constant depending only on p and q. For $q > 2$ we obtain

$$\begin{aligned} d\|v\|_{L^{2p}}^{2pq} &= 2pq\|v\|_{L^{2p}}^{2p(q-1)} d\|v\|_{L^{2p}}^{2p} + 4p^2q(q-1)\|v\|_{L^{2p}}^{2p(q-2)} \left[d\|v\|_{L^{2p}}^{2p}\right]^2 \\ &\leq C\|v\|_{L^{2p}}^{2p(q-1)} \left[-\tfrac{1}{2}\|v\|_{L^{2p}}^{2p} - \tfrac{1}{2}\|v\|_{L^{4p-2}}^{4p-2} - \varepsilon^2\|(v^p)_x\|^2 + \|v\|^2\right] dt \\ &\quad + C\eta_0\|v\|_{L^{2p}}^{2p(q-2)}\|\psi\|^2\, dt + C\|v\|_{L^{2p}}^{2p(q-1)}\langle\psi, dW\rangle. \end{aligned} \tag{5.40}$$

By the estimate of ψ we have for $t \leq \tau^*$

$$\begin{aligned} \eta_0\|v\|_{L^{2p}}^{2p(q-2)}\|\psi\|^2 &\leq C\eta_0\|v\|_{L^{2p}}^{2p(q-1)} \left(\varepsilon\|v^p\|_{H^1}^2 + \varepsilon^{-1}\|v\|_{L^{4p-2}}^{4p-2} + \varepsilon^{-1}\|v\|^2\right) \\ &\leq C\varepsilon^{2m}\|v\|_{L^{2p}}^{2p(q-1)} \left(\varepsilon^2\|v^p\|_{H^1}^2 + \|v\|_{L^{4p-2}}^{4p-2} + \|v\|^2\right). \end{aligned}$$

All those terms are by a magnitude of ε^{2m} smaller than their counterparts in (5.40). This yields

$$\begin{aligned} d\|v\|_{L^{2p}}^{2pq} &\leq -C\|v\|_{L^{2p}}^{2pq}\, dt + C\|v\|_{L^2}^2\|v\|_{L^{2p}}^{2p(q-1)}\, dt + C\|v\|_{L^{2p}}^{2p(q-1)}\langle\psi, dW\rangle \\ &\leq -C\|v\|_{L^{2p}}^{2pq}\, dt + C\varepsilon^{1+2m}\|v\|_{L^{2p}}^{2p(q-1)}\, dt + C\|v\|_{L^{2p}}^{2p(q-1)}\langle\psi, dW\rangle. \end{aligned}$$

From here, we can follow the proof of Theorem 2.14. Inductively we obtain with $\alpha = \varepsilon^{1+2m}$

$$\mathbb{E}\|v(\tau^*)\|_{L^{2p}}^{2pq} \leq C\alpha^q\left[1 + T_\varepsilon\right].$$

By assumption on the L^{2p}-radius R_ε, we have $\alpha/R_\varepsilon^p = \mathcal{O}(\varepsilon^\kappa)$ and the argument can be closed via Chebyshev's inequality. □

Remark 5.49. In the analysis of the stochastic ODE governing the motion of kinks, we had to control the L^{2p}-norm only in Lemma 5.22 when we dealt with bounding the full operator $\mathcal{L}(u^h+v)$. Let τ^* be the stopping time defined by (5.39). For $t \leq \tau^*$ we obtain

$$\langle\mathcal{N}^h(v), v\rangle \leq C\varepsilon^{-1}\|\mathcal{N}^h(v)\|_{L^1} \leq C\varepsilon^{-1}\left[\|v\|_{L^{2p}}^{2p} + \|v\|_{L^2}^2\right] \leq \varepsilon^{1+2m-2\kappa}.$$

This is—as one would expect by the scaling of the radius—exactly the same estimate as in Lemma 5.22. Hence, the analysis of the stochastic ODE remains unaffected by the more general nonlinearity f.

So far, we studied the general case for the stochastic Allen–Cahn equation. The mass conserving version can be treated in a similar fashion. The scaling between the L^2-radius R_ε and the radius with respect to the L^{2p}-norm follows the same rule as before. Therefore, we expect the following result to hold true, but omit the details here.

Conjecture 5.50 (L^4-Stability for (mAC))**.**
For $m > 0$ consider the stopping time

$$\tau^* = \inf\left\{t \in [0, T_\varepsilon \wedge \tau_0] \,:\, \|v(t)\| > \varepsilon^{3/2+m} \text{ or } \|v(t)\|_{L^{2p}} > \varepsilon^{3/2p+m/p-\kappa}\right\},$$

where $T_\varepsilon = \varepsilon^{-M}$ for fixed large $M > 0$. Suppose that for some $\nu \in (0,1)$

$$\|v(0)\| \leq \nu\varepsilon^{3/2+m}, \quad \|v(0)\|_{L^4} \leq \nu\varepsilon^{3/2p+m/p-\kappa}, \quad \text{and} \quad \eta_0 \leq \varepsilon^{4+2m}.$$

Then, the probability $\mathbb{P}(\tau^* < T_\varepsilon \wedge \tau_0)$ is smaller than any power of ε, as ε tends to zero.

APPENDIX A

Basic tools

This chapter is a brief collection of estimates from basic calculus, PDE theory, and functional analysis that were frequently used throughout this thesis. These can be found for example in [Bre11, Eva10]. For a collection of basic tools and inequalities from stochastic analysis, see the following chapter, Appendix B.

We start with one of the basic tools from calculus, namely Young's inequality for products. It can be used to prove Hölder's inequality. In our work, it is widely used to estimate products of different norms by the sum of the same terms with adequately scaled powers.

Theorem A.1 (Young's inequality).
If $a, b \geq 0$ and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. The map $x \mapsto \exp(x)$ is convex and consequently,

$$ab = \exp\left(\tfrac{1}{p}\log(a^p) + \tfrac{1}{q}\log(b^q)\right) \leq \tfrac{1}{p}\exp\left(\log(a^p)\right) + \tfrac{1}{q}\exp\left(\log(b^q)\right) = \tfrac{a^p}{p} + \tfrac{b^q}{q}. \qquad \square$$

If we set $a := (\varepsilon p)^{1/p} a$ and $b := (\varepsilon p)^{-1/p}\, b$, we obtain a scaled version of Young's inequality, also called „Young's inequality with ε“, which allows us to weight the factors differently.

Corollary A.2 (Young's inequality with ε).
If $a, b \geq 0$, $\varepsilon > 0$, and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \varepsilon a^p + \frac{(p\varepsilon)^{1-q}}{q} b^q.$$

In the following theorems, we state some important tools from the theory of Sobolev spaces. One of the fundamental inequalities is the Poincaré inequality.

Theorem A.3 (Poincaré inequality).
Let Ω be a bounded, connected and open subset of $\mathbb{R}^d$ with C^1-boundary and $p \in [1, \infty)$. There exists a constant $C = C(d, p, \Omega) > 0$ such that for every $f \in W^{1,p}(\Omega)$

$$\left\| f - \frac{1}{|\Omega|} \int_\Omega f(x)\,\mathrm{d}x \right\|_{L^p(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)}.$$

Proof. For a proof see [Eva10] Theorem 1 in Section 5.8. $\square$

A prominent role in our study play inclusions between different Sobolev spaces. Given the space $W^{k,p}(\Omega)$ of (weakly) differentiable functions, whose first k weak derivatives are in L^p, the Sobolev embedding theorem is concerned with the question on when there exist continuous inclusions $W^{k,p} \subset W^{l,q}$.

Theorem A.4 (Sobolev embedding).
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Suppose that $k > \ell$ and $1 \leq p < q < \infty$ are such that

$$\frac{1}{p} - \frac{k}{d} = \frac{1}{q} - \frac{\ell}{d}.$$

Then, we have the continuous embedding $W^{k,p}(\Omega) \subset W^{l,q}(\Omega)$. In the special case $k = 1$ and $\ell = 0$, this yields $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, where p^* is the Sobolev conjugate of p given by $p^* = dp/(d-p)$.

Proof. The special case $k = 1, \ell = 0$ is a direct consequence of the Sobolev–Nirenberg–Gagliardo inequality. For the general case see [Eva10], Chapter 5.6, Theorem 6. □

Next is the Gagliardo–Nirenberg interpolation inequality, which interpolates the L^p-norm of a weak derivative.

Theorem A.5 (Gagliardo–Nirenberg interpolation inequality).
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $1 \leq q, r \leq \infty$ and $m \in \mathbb{N}$ be such that for $j \in \mathbb{N}$ and $j/m \leq \alpha \leq 1$

$$\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\alpha + \frac{1-\alpha}{q}.$$

There exists a constant $C > 0$ such that for $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$

$$\|D^j u\|_{L^p(\Omega)} \leq C\|u\|_{L^q(\Omega)}^{1-\alpha}\|u\|_{W^{m,r}(\Omega)}^{\alpha}.$$

Proof. The statement was originally proved by L. Nirenberg [Nir59, p.125]. Some applications can be found in [Bre11]. □

An important special case of this estimate is Ladyzhenskaya's inequality, which is concerned with bounding the L^4-norm of a function in terms of the L^2- and H^1-norm.

Corollary A.6 (Ladyzhenskaya's inequality).
For $d \in \{2, 3\}$ let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $u \in H_0^1(\Omega)$. There exists a constant $C > 0$ such that for $d = 2$

$$\|u\|_{L^4} \leq C\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}.$$

In the three-dimensional case, one obtains

$$\|u\|_{L^4} \leq C\|u\|_{L^2}^{1/4}\|u\|_{H^1}^{3/4}.$$

Agmon's inequality is concerned with controlling the uniform norm of a Sobolev function by means of the Sobolev spaces H^s. For a proof, we refer to the original work of S. Agmon [Agm10, Lemma 13.2].

Theorem A.7 (Agmon's inequality).
Let $\Omega \subset \mathbb{R}^d$ and $s_1 < \frac{d}{2} < s_2$. If $0 < \theta < 1$ and $\frac{d}{2} = \theta s_1 + (1-\theta)s_2$, we have for any $u \in H^{s_2}(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{H^{s_1}(\Omega)}^{\theta}\|u\|_{H^{s_2}(\Omega)}^{1-\theta}.$$

The final results concern fractional Sobolev spaces $W^{s,p}(\Omega)$ (see [DNPV12] for an overview). For $\Omega \subset \mathbb{R}^d$, $0 < s < 1$ and $1 \leq p < \infty$ we define

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{d/p+s}} \in L^p(\Omega \times \Omega) \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}.$$

Similarly to the classical Sobolev spaces, we have the following interpolation inequality for fractional Sobolev spaces.

Theorem A.8 (Interpolation inequality)**.**
Let $0 < s_1 < s_2 < 1$ and $1 < p_0, p_1 < \infty$. Moreover, let $0 < \theta < 1$ be such that

$$s = \theta s_1 + (1-\theta) s_2 \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

Then we have $W^{s,p}(\Omega) \subset W^{s_0,p_0}(\Omega) \cap W^{s_1,p_1}(\Omega)$ and

$$\|u\|_{W^{s,p}} \leq C \|u\|^{\theta}_{W^{s_1,p_1}} \|u\|^{1-\theta}_{W^{s_2,p_2}}$$

We conclude this chapter with a result on fractional embedding. The fractional Sobolev space $W^{s,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for any $q \in [p, p^*]$, where $p^* = \frac{dp}{d-sp}$ is the fractional critical exponent.

Theorem A.9 (Fractional embedding)**.**
Let $\Omega \subset \mathbb{R}^d$ be an open set of class $C^{0,1}$ with bounded boundary. Let $s \in (0,1)$ and $p \in [1, \infty)$ such that $sp < d$. Then, there exists a positive constant $C = C(d, p, s, \Omega)$ such that for any $u \in W^{s,p}(\Omega)$ and any $q \in [p, p^*]$

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)},$$

where $p^* = dp/(d - sp)$ is the fractional critical exponent. If, in addition, the set Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for any $q \in [1, p^*]$.

APPENDIX B

Preliminaries from stochastic analysis

We collect some important definitions and theorems from stochastic analysis. In the first part, we give the definitions of a $\mathcal{Q}$-Wiener process and stochastic integration. Thereafter, we discuss semigroups and stochastic differential equations. Under Lipschitz conditions we state a result on existence and uniqueness of solutions. Throughout these sections, we mainly follow [DPZ92b]. The final part is devoted to some inequalities from stochastic analysis.

B.1 $\mathcal{Q}$-Wiener processes and stochastic integration

Throughout this section, $\mathcal{H}$ and $\mathcal{K}$ denote separable Hilbert spaces, $L(\mathcal{H},\mathcal{K})$ the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, and $\mathrm{HS}(\mathcal{H},\mathcal{K})$ all Hilbert–Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$. Moreover, we fix an abstract probability space $(\Omega,\mathcal{F},\mathbb{P})$ together with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$.

Definition B.1 ($\mathcal{Q}$-Wiener process)**.**
We call an $\mathcal{H}$-valued stochastic process $\{W(t)\}_{t\geq 0}$ a *$\mathcal{Q}$-Wiener process*, if

- $W(0)=0$ ($\mathbb{P}$-almost sure),
- W has $\mathbb{P}$-almost sure continuous trajectories,
- The increments of W are independent, i.e., the random variables

  $$W(t_1)\,,\, W(t_2)-W(t_1)\,,\,\dots\,,\, W(t_n)-W(t_{n-1})$$

  are independent for any choice $n\in\mathbb{N}$ and $0\leq t_1\leq t_2\leq\ldots\leq t_n$,
- The increments are Gaussian with $W(t)-W(s)\sim\mathcal{N}(0,(t-s)\mathcal{Q})$ for all $0\leq s\leq t$.

Crucial for our analysis are covariance operators that are of trace-class.

Definition B.2 (Trace-class operators)**.**
A non-negative operator $\mathcal{Q}\in L(\mathcal{H})$ is *of trace-class*, if for an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$

$$\mathrm{trace}(\mathcal{Q}) \;=\; \sum_{k\in\mathbb{N}}\langle \mathcal{Q}e_k,e_k\rangle_{\mathcal{H}} \;<\; \infty.$$

Let $u\in\mathcal{H}$ with $\|u\|=1$. By completing u to an orthonormal basis of $\mathcal{H}$ we see that $\langle\mathcal{Q}u,u\rangle\leq\mathrm{trace}(\mathcal{Q})$ and thus we always have $\|\mathcal{Q}\|_{L(\mathcal{H})}=\sup_{u\in\mathcal{H},\|u\|=1}\langle\mathcal{Q}u,u\rangle\leq\mathrm{trace}(\mathcal{Q})$. Also, note that every trace-class operator is compact and therefore, by the spectral theorem for compact operators, there exists a basis of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$ and eigenvalues α_k^2, i.e., $\mathcal{Q}e_k=\alpha_k^2e_k$.

Theorem B.3 (Series expansion).
Let $\mathcal{Q}$ be a trace-class operator with orthonormal basis of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$ and eigenvalues $\{\alpha_k^2\}_{k\in\mathbb{N}}$. If W is a $\mathcal{Q}$-Wiener process, then it is given by the series expansion

$$W(t) \;=\; \sum_{k\in\mathbb{N}} \alpha_k \beta_k(t) e_k$$

for a family $\{\beta_k\}_{k\in\mathbb{N}}$ of independent real-valued standard Brownian motions.

We continue with the definition of the stochastic integral with respect to a $\mathcal{Q}$-Wiener process for a broad class of stochastic processes. We start with defining the integral for elementary processes and then, extend it via completion and localization to a more general class of stochastic processes.

Definition B.4 (Elementary Processes).
We call a stochastic process $\Phi(t),\ t \in [0,T]$ *elementary* if there exist $0 = t_0 < \ldots < t_n = T$, $n \in \mathbb{N}$ and $\mathcal{F}_{t_{k-1}}$-measurable random variables $\Phi_k : \Omega \to L(\mathcal{H},\mathcal{K})$ such that for any $t \in [0,T]$

$$\Phi(t) \;=\; \sum_{k=1}^{n} \Phi_k \chi_{(t_{k-1},t_k)}(t).$$

In the sequel, we denote the space $L^2([0,T]\times\Omega, \mathrm{HS}(\mathcal{Q}^{1/2}(\mathcal{H}),\mathcal{K}))$ by $\mathcal{L}_T$.

Definition B.5 (Stochastic integral for elementary processes).
Let $\Phi \in \mathcal{L}_T$ be an elementary process. We define the $\mathcal{K}$-valued *stochastic integral*

$$\int_0^T \Phi(s)\,\mathrm{d}W(s) \;:=\; \sum_{k=1}^{n} \Phi_k \left(W(t_k) - W(t_{k-1})\right).$$

Via Itô isometry, the map induced by the stochastic integral is (for elementary processes) an isometry between the spaces $\mathcal{L}_T$ and $L^2(\Omega,\mathcal{K})$.

Proposition B.6 (Itô isometry).
If $\Phi \in \mathcal{L}_T$ is an elementary process, then

$$\mathbb{E}\Big\| \int_0^T \Phi(s)\,\mathrm{d}W(s)\Big\|_{\mathcal{K}}^2 \;=\; \mathbb{E}\left(\int_0^T \|\Phi(s)\circ\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s\right).$$

We extend the stochastic integral via approximation by elementary processes to a broader class of stochastic processes. If X is a predictable process in $\mathcal{L}_T$, then there exists a sequence of elementary processes $\{X_n\}_{n\in\mathbb{N}}$ such that $X_n \to X$ in $\mathcal{L}_T$. Hence, the stochastic integral can be defined on the subset $\mathcal{N}_w^2(0,T) \subset \mathcal{L}_T$ of predictable processes, i.e.,

$$\mathcal{N}_w^2(0,T) \;=\; \Big\{\Phi : [0,T]\times\Omega \to \mathrm{HS}(\mathcal{Q}^{1/2}(\mathcal{H}),\mathcal{K}) \;:\; \Phi \in \mathcal{L}_T \text{ and } \Phi \text{ is predictable}\Big\}.$$

Via localization the stochastic integral can be extended further to the linear space $\mathcal{N}_w(0,T)$ given by

$$\mathcal{N}_w(0,T) \;=\; \Big\{\Phi : [0,T]\times\Omega \to \mathrm{HS}(\mathcal{Q}^{1/2}(\mathcal{H}),\mathcal{K}) \;:\; \Phi \text{ is predictable and } \mathbb{P}\left(\int_0^T \|\Phi(s)\circ\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s \;<\; \infty\right) \;=\; 1\Big\}.$$

An important role in the study of stochastic differential equations plays Itô's formula. We give conditions on a function $F : [0,T] \times \mathcal{K} \to \mathbb{R}$ under which $F(t, X(t))$ has a stochastic differential provided that X has a stochastic differential. The extension to $\mathbb{R}^n$-valued functions is straightforward and omitted.

Theorem B.7 (Itô formula)**.**
Let $\mathcal{Q}$ be a symmetric non-negative trace-class operator on a separable Hilbert space $\mathcal{H}$ and $\{W(t)\}_{t\in[0,T]}$ a $\mathcal{Q}$-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. Assume that a stochastic process $X(t)$, $0 \le t \le T$, is given by

$$X(t) \;=\; X(0) + \int_0^t \Psi(s)\,\mathrm{d}s + \int_0^t \Phi(s)\,\mathrm{d}W(s),$$

where $X(0)$ is a $\mathcal{F}_0$-measurable $\mathcal{K}$-valued random variable, $\Psi(s)$ is a $\mathcal{K}$-valued $\mathcal{F}_s$-measurable process with

$$\int_0^T \|\Psi(s)\|_{\mathcal{K}}\,\mathrm{d}s < \infty \quad \mathbb{P}\text{-almost sure},$$

and $\Phi \in \mathcal{N}_w(0,T)$. Furthermore, assume that a function $F : [0,T] \times \mathcal{K} \to \mathbb{R}$ is such that F is continuous and its Fréchet derivatives F_t, F_x, F_{xx} are continuous and bounded on bounded subsets of $[0,T] \times \mathcal{K}$. Then, the following Itô formula holds true $\mathbb{P}$-almost sure for all $t \in [0,T]$

$$\begin{aligned} F(t, X(t)) \;=\; & F(0, X(0)) + \int_0^t \langle F_x(s, X(s)), \Phi(s)\,\mathrm{d}W(s)\rangle_{\mathcal{K}} \\ & + \int_0^t F_t(s, X(s)) + \langle F_x(s, X(s)), \Psi(s)\rangle_{\mathcal{K}}\,\mathrm{d}s \\ & + \frac{1}{2}\int_0^t \operatorname{trace}\Big(F_{xx}(s, X(s))(\Phi(s)\mathcal{Q}^{1/2})(\Phi(s)\mathcal{Q}^{1/2})^*\Big)\,\mathrm{d}s. \end{aligned}$$

B.2 Semigroups and stochastic PDEs

Here, we give the definition of a mild solution to a stochastic PDE. First, we recall some basic definitions from semigroup theory. For more details we refer to [BB67, Paz83, Yos80].

Definition B.8 (C_0-semigroup)**.**
A family $S(t) \in L(\mathcal{K})$, $t \ge 0$, of bounded linear operators is called C_0-*semigroup* if

- $S(0) = I$
- $S(t+s) \;=\; S(t)S(s) \quad \forall t, s \ge 0$
- $\lim_{t\to 0^+} S(t)x \;=\; x \quad \forall x \in \mathcal{K}$

Definition B.9 (Infinitesimal generator of a semigroup)**.**
Let $S(t)$ be a C_0-semigroup. The linear operator A with domain

$$\mathcal{D}(A) \;=\; \left\{x \in \mathcal{K} \,:\, \lim_{t\to 0^+} \frac{S(t)x - x}{t} \;\text{ exists}\right\}$$

defined by

$$Ax \;=\; \lim_{t\to 0^+} \frac{S(t)x - x}{t}$$

is called *infinitesimal generator* of the semigroup $S(t)$.

We can now give a meaning to the stochastic PDE

$$\begin{cases} dX(t) &= (AX(t) + F(t, X(t)))\, dt + B(t, X(t))\, dW(t) \\ X(0) &= X_0. \end{cases} \tag{B.1}$$

Here, $A : \mathcal{D}(A) \subset \mathcal{K} \mapsto \mathcal{K}$ is the generator of a C_0-semigroup, $W(t)$ denotes a $\mathcal{Q}$-Wiener process, and the initial condition X_0 is assumed to be an $\mathcal{F}_0$-measurable $\mathcal{K}$-valued random variable. The coefficients F and B are in general given by nonlinear maps

$$F : \Omega \times [0,T] \times C([0,T], \mathcal{K}) \mapsto \mathcal{K} \quad \text{and} \quad B : \Omega \times [0,T] \times C([0,T], \mathcal{K}) \mapsto \mathrm{HS}(\mathcal{Q}^{1/2}(\mathcal{H}), \mathcal{K}).$$

Definition B.10 (Mild solution)**.**
A stochastic process $X(t)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in[0,T]}, \mathbb{P})$ and adapted to the filtration $\{\mathcal{F}\}_{t\in[0,T]}$ is called a *mild solution* to (B.1), if

$$\mathbb{P}\left(\int_0^T \|X(t)\|_{\mathcal{K}}\, \mathrm{d}t \,<\, \infty\right) \,=\, 1,$$

$$\mathbb{P}(\left(\int_0^T \|F(t, X(t))\|_{\mathcal{K}} + \|B(t, X(t))\|^2_{\mathrm{HS}(\mathcal{Q}^{1/2}(\mathcal{H}),\mathcal{K})}\, \mathrm{d}t \,<\, \infty\right) \,=\, 1,$$

and for all $t \le T$ ($\mathbb{P}$-almost sure)

$$X(t) \,=\, S(t)X_0 + \int_0^t S(t-s)F(s, X(s))\, \mathrm{d}s + \int_0^t S(t-s)B(s, X(s))\, \mathrm{d}W(s).$$

Remark B.11 (Conversion into a Stratonovich integral)**.**
By Itô formula (Theorem B.7), the chain rule does not hold true for stochastic differentials. One way to overcome this problem is to define the stochastic integral in a different way, namely the Stratonovich integral. Let $X(t)$ be a mild solution of (B.1) and $\Phi : [0,T] \times \mathcal{K} \to L(\mathcal{H}, \mathcal{K})$ a sufficiently smooth operator. Then, one can make sense of the following definition:

$$\int_0^t \Phi(s, X(s)) \circ \mathrm{d}W \,:=\, \int_0^t \Phi(s, X(s))\, \mathrm{d}W + \frac{1}{2}\int_0^t \mathrm{trace}(\mathcal{Q}\, D_2\Phi(s, X(s))\, B(s, X(s))\, \mathrm{d}s.$$

For a more detailed prescription and exact conditions we refer to [TN04]. Basically, the Itô correction term is absorbed into the definition of the Stratonovich integral. The price one has to pay though is that the Stratonovich integral no longer is a martingale.

Under certain measurability and Lipschitz conditions on the coefficients F and B, one can show that there exists a unique solution to (B.1). We assume that the following conditions hold true:

(S1) F and B are jointly measurable, and for every $0 \le t \le T$ they are measurable with respect to the product field $\mathcal{F}_t \otimes \mathcal{G}_t$ on $\Omega \times C([0,T], \mathcal{K})$, where $\mathcal{G}_t$ is a σ-field generated by cylinders with basis over $[0,t]$.

(S2) There exists a constant $c > 0$ such that for all $x \in C([0,T], \mathcal{K})$

$$\|F(\omega, t, x)\|_{\mathcal{K}} + \|B(\omega, t, x)\|_{\mathrm{HS}(\mathcal{Q}^{1/2}(\mathcal{H}),\mathcal{K})} \,\le\, c\left(1 + \sup_{0\le t\le T} \|x(t)\|_{\mathcal{K}}\right).$$

(S3) For all $x, y \in C([0,T], \mathcal{K})$ there exists a constant $C > 0$ such that

$$\|F(\omega, t, x) - F(\omega, t, y)\|_{\mathcal{K}} + \|B(\omega, t, x) - B(\omega, t, y)\|_{\mathrm{HS}(\mathcal{Q}^{1/2}(\mathcal{H}),\mathcal{K})} \le C \sup_{0\le s\le T} \|x(s) - y(s)\|_{\mathcal{K}}.$$

Theorem B.12 (Existence and uniqueness)**.**
Let F and B satisfy (S1)-(S3). Then (B.1) has an unique continuous mild solution. Moreover, if $\mathbb{E}\|X_0\|_{\mathcal{K}}^{2p} < \infty$ for some $p > 1$, then the solution $X(t)$ satisfies

$$\mathbb{E} \sup_{0 \le s \le T} \|X(s)\|_{\mathcal{K}}^{2p} < \infty.$$

In the case $p = 1$, we have to assume that $S(t)$ is a pseudo-contraction semigroup, that is, a semigroup such that for some $\alpha \ge 0$ we have $\|S(t)\|_{L(\mathcal{K})} \le \exp(\alpha t)$ for any $t \ge 0$.

B.3 Basic tools from stochastic analysis

We close this brief review of stochastic analysis with some basic tools that were used throughout this work.

Theorem B.13 (Chebyshev's inequality, [Tch67])**.**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}_0^+$ a random variable. Then, for any $R > 0$ and $0 < p < \infty$

$$\mathbb{P}(X \ge R) \le \frac{\mathbb{E}[X^p]}{R^p}.$$

Proof.

$$\mathbb{E}[X^p] = \int_\Omega X^p \, \mathrm{d}\mathbb{P} \ge \int_{\{\omega : X(\omega) \ge R\}} X^p \, \mathrm{d}\mathbb{P} \ge \int_{\{\omega : X(\omega) \ge R\}} R^p \, \mathrm{d}\mathbb{P} = R^p \, \mathbb{P}(X \ge R). \qquad \square$$

The following theorem introduces the Burkholder–Davis–Gundy inequality, which deals with bounding the expectation of the supremum of a local martingale up to a stopping time. Note that the stochastic integral defined in Section B.1 is a (local) martingale. We take care of this special case in Corollary B.15, where the constant is explicitly given. We denote the quadratic variation of a stochastic process $\{X(t)\}_{t \in [0,T]}$ by $[X]_t$, that is,

$$[X]_t = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \left(X(t_k) - X(t_{k-1})\right)^2,$$

where $\|P\|$ is the mesh size over partitions of the interval $[0, t]$. The following inequality was first proved in [BDG72].

Theorem B.14 (Burkholder–Davis–Gundy inequality)**.**
Let X be a local martingale with $X_0 = 0$ and τ an $\mathcal{F}$-stopping time. Then, for any $1 \le p < \infty$, there exist positive constants c_p, C_p such that the following inequality holds

$$c_p \mathbb{E}\left[[X]_\tau^{p/2}\right] \le \mathbb{E}\left(\sup_{s \le \tau} |X_s|\right)^p \le C_p \mathbb{E}\left[[X]_\tau^{p/2}\right].$$

Corollary B.15 (Burkholder–Davis–Gundy for stochastic integrals)**.**
For $p \ge 1$ and $\Phi(t) \in \mathrm{HS}(\mathcal{Q}^{1/2}(\mathcal{H}), \mathcal{K})$, $t \in [0, T]$, we have

$$\mathbb{E} \sup_{0 \le s \le T} \left\| \int_0^s \Phi(s) \, \mathrm{d}W(s) \right\|_{\mathcal{K}}^{2p} \le C_p \mathbb{E} \left(\int_0^T \|\Phi(s) \circ \mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2 \, \mathrm{d}s \right)^p,$$

where the constant C_p is explicitly given by $C_p = (p(2p-1))^p \left(\frac{2p}{2p-1}\right)^{2p^2}$.

Finally, we state Doob's optional sampling theorem in a for the purpose of this work abbreviated version. Basically, properties of martingales generalize to stopping times. For a proof we refer to [Kle06].

Theorem B.16 (Doob's optional stopping theorem).
Let $\{X_t\}_{t\geq 0}$ be a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Suppose that τ is a stopping time such that there exists a positive integer N with $\tau(\omega) \leq N$ for all $\omega \in \Omega$. Then, X_τ is integrable, and satisfies

$$\mathbb{E}X_\tau = \mathbb{E}X_0.$$

APPENDIX C

Existence, uniqueness and regularity of solutions to the stochastic Cahn–Hilliard equation

This chapter is devoted to establishing existence, uniqueness and regularity of solutions to the stochastic Cahn–Hilliard and Allen–Cahn equation in the relevant space dimensions. Here, we will only treat the stochastic Cahn–Hilliard equation in the three-dimensional case, but the adaption to other space dimensions or the Allen–Cahn equation is straightforward. Moreover, for simplicity, the parameter $\varepsilon > 0$ is scaled out and thus we consider $\varepsilon = 1$. We state some results from [DPD96]. For more details on the stochastic Cahn–Hilliard equation, we refer to [CW01, EM91] and for a general overview to [DPZ96, DPZ92b, GP93, Zab89].

Let us start by introducing some notation. For a smooth bounded domain $\Omega \subset \mathbb{R}^3$ we consider $L^2(\Omega)$ with the standard norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. The subspace of functions with mean zero is denoted by

$$L_0^2(\Omega) \,=\, \left\{ u \in L^2(\Omega) \,:\, m(u) \,:=\, \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x \,=\, 0 \right\}.$$

For $u \in L^2(\Omega)$, the projection onto $L_0^2(\Omega)$ is denoted by $\bar{u} := u - m(u)$. On $L_0^2(\Omega)$ we define the linear unbounded operator A by

$$Au \,=\, -\Delta u, \quad u \in D(A) \,=\, \left\{ u \in H^2(\Omega) \,:\, \frac{\partial u}{\partial \eta} \,=\, 0 \right\}.$$

The operator A is self-adjoint, positive and has a compact resolvent. Furthermore, there exists an orthonormal basis in $L^2(\Omega)$ of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$ with corresponding eigenvalues

$$0 \,=\, \lambda_0 \,<\, \lambda_1 \,\le\, \lambda_2 \,\le\, \ldots \,\le\, \lambda_k \to \infty.$$

Also, note that the function e_0 is constant and equal to $|\Omega|^{-1/2}$. For $s \in \mathbb{R}$ and $u = \sum_j \alpha_j e_j$, we define the fractional power $A^{s/2}$ by

$$A^{s/2}u \,:=\, \sum_{j\in\mathbb{N}} \lambda_j^{s/2}\alpha_j e_j$$

with domain $\mathcal{H}^s := D(A^{s/2}) = \{u = \sum_{j\in\mathbb{N}} \alpha_j e_j \,:\, \sum_{j\in\mathbb{N}} \lambda_j^s \alpha_j^2 < \infty\}$. We endow $\mathcal{H}^s$ with the seminorm and semiscalar product

$$|u|_s \,:=\, \|A^{s/2}u\|, \quad (u,v)_s \,:=\, \langle A^{s/2}u, A^{s/2}v\rangle$$

and the norm

$$\|u\|_s \,:=\, \left(|u|_s^2 + m(u)^2\right)^{1/2}.$$

In this setting, the stochastic Cahn–Hilliard equation is then given by

$$du \,=\, \left[-A^2u - Af(u)\right]\mathrm{d}t + dW, \quad u(0) \,=\, u_0. \tag{CH}$$

For simplicity of presentation, we will always assume that $f(u) = u^3 - u$ in the sequel. In [DPD96], polynomials of odd degree with positive leading coefficient were treated. Moreover, W denotes a $\mathcal{Q}$-Wiener process and is given by the series expansion

$$W(t) = \sum_{k\in\mathbb{N}} \alpha_k \beta_k(t) f_k,$$

where $\{f_k\}_{k\in\mathbb{N}}$ forms an orthonormal basis of eigenfunctions of $\mathcal{Q}$, i.e., $\mathcal{Q}f_k = \alpha_k^2 f_k$, and $\{\beta_k\}_{k\in\mathbb{N}}$ is a family of independent real-valued standard Brownian motions. Define the stochastic convolution

$$W_A(t) := \int_0^t e^{-(t-s)A^2} \,\mathrm{d}W(s),$$

which is the unique solution to the linear problem

$$\begin{cases} du + A^2 u\, dt = dW, \\ u(0) = 0. \end{cases}$$

Note that, in the case of a cylindrical Wiener process, W_A has an α-Hölder continuous version for any exponent $\alpha \in [0, 1/8)$. For a smoother noise, i.e., $\mathrm{trace}(A^{1-\delta}\mathcal{Q}) < \infty$ for some $\delta > 0$, ∇W_A has an α-Hölder continuous version for any exponent $\alpha \in [0, \delta/4)$.

If $u(t)$ is a solution to (CH), one easily obtains that $v(t) = u(t) - W_A(t)$ is a solution to

$$\begin{cases} \frac{dv}{dt} + A^2 v + Af(v + W_A) = 0, \\ v(0) = u_0. \end{cases} \tag{C.1}$$

First, let us show that solutions to (C.1), or equivalently (CH), are unique. Crucial for establishing uniqueness is the differentiability of the $\mathcal{H}^{-1}$-norm. Lemma 2.1 in [DPD96] gives a criteria for achieving that. A proof of an easier version can be found for example in [Eva10, Section 5.9, Theorem 3].

Lemma C.1. *Let $u \in L^2([0,T];\mathcal{H}^{-1}) \cap L^4([0,T]\times\Omega)$ be such that*

$$A^{-1}\,\partial_t \bar{u} = w_1 + w_2 \quad \textit{with} \quad w_1 \in L^2([0,T];\mathcal{H}^{-1}) \;\; \textit{and} \;\; w_2 \in L^{4/3}([0,T]\times\Omega).$$

If the map $t \mapsto m(u(t))$ is continuous, then $u \in C([0,T];\mathcal{H}^{-1})$ and $\partial_t |u|_{-1}^2 = 2\langle \bar{u}, A^{-1}\partial_t \bar{u}\rangle$.

With the help of this lemma, one can finally prove uniqueness of solutions to (C.1). The main idea is to apply Gronwall's lemma to the difference of two solutions.

Theorem C.2. *Let $u_0 \in \mathcal{H}^{-1}$. Then, there exists at most one solution to (CH) that lies in $L^2([0,T];\mathcal{H}^1) \cap L^4([0,T]\times\Omega)$.*

Proof. Let $u, v \in L^2([0,T];\mathcal{H}^1) \cap L^4([0,T]\times\Omega)$ be solutions to (C.1) and define $z := u - v$. Then, z solves

$$\begin{cases} \partial_t z + A^2 z + Af(u + W_A) - Af(v + W_A) = 0, \\ z(0) = u_0 - v_0 = 0. \end{cases}$$

Note that the conservation of mass obviously implies that $m(z(t)) = 0$ for all $t \geq 0$. Moreover, $Az \in L^2([0,T];\mathcal{H}^{-1})$ and, by the regularity of W_A and f being a cubic polynomial, $f(u + W_A) - f(v + W_A) \in L^{4/3}([0,T]\times\Omega)$. Hence, Lemma C.1 is applicable and yields

$$\partial_t |z|_{-1}^2 = 2\langle z, A^{-1}\partial_t z\rangle = -2|z|_1^2 + 2\langle f(v + W_A) - 2f(v + z + W_A), z\rangle.$$

For the last term we obtain via Hölder's inequality

$$\begin{aligned}\langle f(v+W_A)-f(v+z+W_A),z\rangle \;&=\; \|z\|_{L^2}^2-\|z\|_{L^4}^4-\int_\Omega z^2(v+W_A)^2+z^3(v+W_A)\,\mathrm{d}x\\ &\le\; \|z\|_{L^2}^2-\frac{1}{2}\|z\|_{L^4}^4-\frac{1}{2}\int_\Omega z^2(v+W_A)^2\,\mathrm{d}x.\end{aligned}$$

We now use $\|z\|_{L^2}^2 \le |z|_{-1}|z|_1 \le \frac{1}{2}|z|_{-1}^2+\frac{1}{2}|z|_1^2$ and derive

$$\partial_t|z|_{-1}^2 \;\le\; -2|z|_1^2+2\|z\|_{L^2}^2 \;\le\; |z|_{-1}^2.$$

Applying Gronwall's lemma yields $|z|_{-1}^2=0$, and thus uniqueness. □

As a next step, we investigate the existence of solutions. Under certain regularity assumptions on the $\mathcal{Q}$-Wiener process W and the initial condition u_0, the solution of (C.1) is constructed via Galerkin approximation. For this purpose, we denote for $m\in\mathbb{N}$ the orthogonal projection on $\operatorname{span}\{e_0,\dots,e_m\}$ by P_m and set $v_m := P_m v$ and $W_A^m := P_m W_A$. The Galerkin approximation of (C.1) is then given by

$$\begin{cases}\partial_t\, v_m + A^2 v_m + P_m A f(v_m+W_A^m) \;=\; 0,\\ v_m(0) \;=\; P_m u_0.\end{cases} \tag{C.2}$$

We state the first result on existence for a cylindrical Wiener process, that is $\mathcal{Q}\equiv I$, and an initial value in $\mathcal{H}^{-1}$ (cf. [DPD96], Section 2.1). Note that it is clear that Theorem C.3 still holds true if we replace the cylindrical Wiener process by smoother noise.

Theorem C.3. *Assume that $\mathcal{Q}\equiv I$ and u_0 is $\mathcal{F}_0$-measurable with values in $\mathcal{H}^{-1}$. Then, there exists a unique solution $u(t)$ of the Cahn–Hilliard equation (CH) with $u\in C([0,T];\mathcal{H}^{-1})$.*

Proof. By taking the semiscalar product of (C.2) with v_m, we obtain

$$\frac{1}{2}\partial_t|v_m|_{-1}^2+|v_m|_1^2+\langle P_m f(v_m+W_A^m),\bar{v}_m\rangle \;=\; 0. \tag{C.3}$$

Since the projection P_m is selfadjoint, the last term is given by

$$\langle P_m f(v_m+W_A^m),\bar{v}_m\rangle \;=\; \langle f(v_m+W_A^m),v_m+W_A^m\rangle-\langle f(v_m+W_A^m),m(v_m)+W_A^m\rangle.$$

In our toy case $f(x)=x^3-x$, we have for all $x\in\mathbb{R}$

$$x\cdot f(x) \;\ge\; \frac{1}{2}\left(x^4-1\right) \quad\text{and}\quad |f(x)| \;\le\; 2(|x|^3+1).$$

This implies that

$$\langle f(v_m+W_A^m),v_m+W_A^m\rangle \;\ge\; \frac{1}{2}\|v_m+W_A^m\|_{L^4}^4-\frac{1}{2}|\Omega|,$$

and, by Hölder's and Young's inequality,

$$\begin{aligned}&\langle f(v_m+W_A^m),m(v_m)+W_A^m\rangle\\ &\le\; 2\int_\Omega |v_m+W_A^m|^3\left(|m(v_m)|+|W_A^m|\right)\mathrm{d}x+2\int_\Omega\left(|m(v_m)|+|W_A^m|\right)\mathrm{d}x\\ &\le\; 2\|v_m+W_A^m\|_{L^4}^3\left(|\Omega|^{1/4}|m(v_m)|+\|W_A^m\|_{L^4}\right)+2|\Omega||m(v_m)|+2|\Omega|^{3/4}\|W_A^m\|_{L^4}\\ &\le\; \frac{1}{4}\|v_m+W_A^m\|_{L^4}^4+c\left(|m(v_m)|^4+\|W_A^m\|_{L^4}^4+1\right).\end{aligned}$$

Plugging these two estimates into (C.3) yields

$$\frac{1}{2}\partial_t|v_m|_{-1}^2 + |v_m|_1^2 + \frac{1}{4}\|v_m + W_A^m\|_{L^4}^4 \;\leq\; c\left(|m(v_m)|^4 + \|W_A^m\|_{L^4}^4 + 1\right). \tag{C.4}$$

We observe that $m(v_m(t)) = m(P_m u_0) = m(u_0)$ for all $t \geq 0$. Moreover, by the Hölder regularity of the stochastic convolution W_A, we see that $\|W_A^m\|_{L^4}$ is uniformly bounded. Hence, we conclude that the right-hand side of (C.4) is uniformly bounded and thus the sequence $(v_m)_{m\in\mathbb{N}}$ is bounded in $L^\infty([0,T];\mathcal{H}^{-1})$, $L^2([0,T];\mathcal{H}^1)$, and $L^4([0,T]\times\Omega)$. This implies that

$$Av_m \in L^2([0,T];\mathcal{H}^{-1}) \quad \text{and} \quad f(v_m + W_A^m) \in L^{4/3}([0,T]\times\Omega).$$

The spaces $L^2([0,T];\mathcal{H}^{-1})$ and $L^{4/3}([0,T]\times\Omega)$ are both embedded into $L^{4/3}([0,T];\mathcal{H}^{-2})$ and therefore, we derive that $\partial_t v_m = -A^2 v_m - P_m A f(v_m + W_A^m)$ lies in $L^{4/3}([0,T];\mathcal{H}^{-4})$. By a classical compactness argument (cf. [Aub63]), it can be inferred that (v_m) has a subsequence, which is strongly convergent in $L^2([0,T];\mathcal{H}^0)$ to a function $v \in L^2([0,T];\mathcal{H}^1)\cap L^4([0,T]\times\Omega)$. This shows that

$$A^{-1}\,\partial_t v \in L^2([0,T];\mathcal{H}^{-1}) + L^4([0,T]\times\Omega),$$

and by Lemma C.1 we conclude that v is continuous with values in $\mathcal{H}^{-1}$. Since we already know that solutions to (CH) are unique, the whole sequence $(v_m)_{m\in\mathbb{N}}$ converges to v. □

In our applications, we typically need more regularity of solutions to (CH). To establish that, we automatically have to assume that the noise is even smoother, namely $\operatorname{trace}(A\mathcal{Q}) < \infty$. Note that this condition also implies that the covariance operator $\mathcal{Q}$ is of trace-class. We define the functional $J(u)$ by

$$J(u) \;\coloneqq\; \int_\Omega \left(\frac{1}{2}|\nabla v|^2 + F(u)\right)\,\mathrm{d}x,$$

where F denotes the primitive of f vanishing at zero. It is well known that $J(u)$ is a Lyapunov functional for the deterministic Cahn–Hilliard equation, i.e., $\frac{d}{dt}J(u(t)) \leq 0$ for a solution $u(t)$. In Section 2.3 of [DPD96], this was extended to the stochastic case. Therewith, one can prove $\mathcal{H}^1$-regularity of solutions provided the initial value lies in $\mathcal{H}^1$.

Theorem C.4. *Suppose that* $\operatorname{trace}(A\mathcal{Q}) < \infty$. *Furthermore, assume that* $u_0 \in \mathcal{H}^1$ *is* $\mathcal{F}_0$*-measurable and* $\mathbb{E}J(u_0) < \infty$. *Then,* $u \in C([0,T];\mathcal{H}^1)$ *and*

$$\mathbb{E}J(u(t)) \;\leq\; e^{\operatorname{trace}(\mathcal{Q})t}\left(\mathbb{E}J(u(0)) + \operatorname{trace}(A\mathcal{Q})^{-1}\operatorname{trace}(\mathcal{Q}) + C\right).$$

Proof. For u_m defined by $u_m \coloneqq v_m + W_A^m$ we have

$$du_m + A^2 u_m + P_m A f(u_m) \;=\; P_m\,dW, \quad u_m(0) \;=\; P_m u_0,$$

where $P_m W(t) = \sum_{k=0}^m \alpha_k \beta_k(t) f_k$. Applying Itô formula yields

$$dJ(u_m) \;=\; \langle J_u(u_m), P_m dW\rangle + \langle J_u(u_m), -A^2 u_m - P_m A f(u_m)\rangle\,dt + \tfrac{1}{2}\operatorname{trace}(J_{uu}(u_m)P_m\mathcal{Q})\,dt, \tag{C.5}$$

where J_u and J_{uu} denote derivatives of the functional J. We have $J_u(u_m) = Au_m + f(u_m)$ and therefore

$$\langle J_u(u_m), -A^2 u_m - P_m A f(u_m)\rangle \;=\; -|Au_m + f(u_m)|_1^2. \tag{C.6}$$

For the second derivative one computes $J_{uu}(u_m) = A + f'(u_m)$ and thus

$$\operatorname{trace}(J_{uu}(u_m)P_m\mathcal{Q}) \;=\; \operatorname{trace}(AP_m\mathcal{Q}) + \sum_{k=0}^m \alpha_k^2 \int_\Omega f'(u_m) f_k^2\,\mathrm{d}x.$$

The eigenfunctions f_k can be expressed as a product of sines and cosines times a constant, which only depends on $|\Omega|$. Therefore, we find a constant c such that $\|f_k\|_{L^\infty} \leq c$. Hence we obtain

$$\begin{aligned}\Big|\int_\Omega f'(u_m) f_k^2 \,\mathrm{d}x\Big| &\leq c^2 \int_\Omega |f'(u_m)| \,\mathrm{d}x \leq 3c^2 \int_\Omega |u_m|^2 \,\mathrm{d}x + c^2 \\ &\leq C \int_\Omega |u_m|^4 \,\mathrm{d}x + C \leq C \int_\Omega F(u_m) \,\mathrm{d}x + C \leq J(u_m) + C.\end{aligned}$$

Here we used that $|f'(x)| \leq 3x^2 + 1$ and $F(x) \geq ax^4 - b$ for some positive constants a, b. So far we have shown that

$$\operatorname{trace}(J_{uu}(u_m) P_m \mathcal{Q}) \leq \operatorname{trace}(A P_m \mathcal{Q}) + \operatorname{trace}(P_m \mathcal{Q})(J(u_m) + C). \tag{C.7}$$

Relations (C.5), (C.6), and (C.7) together with $\mathbb{E}\langle J_u(u_m), P_m\, dW\rangle = 0$ now imply

$$\begin{aligned}\frac{d}{dt}\mathbb{E}J(u_m) &= \mathbb{E}\langle J_u(u_m), -A^2 u_m - P_m A f(u_m)\rangle + \frac{1}{2}\operatorname{trace}(J_{uu}(u_m) P_m \mathcal{Q}) \\ &\leq \operatorname{trace}(A P_m \mathcal{Q}) + \operatorname{trace}(P_m \mathcal{Q})(\mathbb{E}J(u_m) + C) \\ &\leq \operatorname{trace}(\mathcal{Q})\mathbb{E}J(u_m) + \operatorname{trace}(A\mathcal{Q}) + C \operatorname{trace}(\mathcal{Q}).\end{aligned}$$

Applying Gronwall's inequality yields

$$\mathbb{E}J(u_m(t)) \leq e^{\operatorname{trace}(\mathcal{Q})t}\Big(\mathbb{E}J(u_m(0)) + \operatorname{trace}(A\mathcal{Q})^{-1}\operatorname{trace}(\mathcal{Q}) + C\Big).$$

If $\mathbb{E}J(u_0) < \infty$, the claim is verified via lower semicontinuity. $\square$

Remark C.5 (Local existence)**.**
The preceding results provided us with a global solution to the stochastic Cahn–Hilliard equation. For the purpose of our work though, local solutions are sufficient. Let us—on a heuristic level—show that for an initial value $u_0 \in \mathcal{H}^2$ we find a positive time $t_0 > 0$ such that the solution lies in $L^\infty([0, t_0], \mathcal{H}^2)$. We expect that it is possible to show global existence, but the technical details are rather delicate. Also, note that it is straightforward to extend this local result to fractional Sobolev spaces of higher order. Similarly to the proof of Theorem C.3 we obtain

$$\frac{1}{2}\partial_t |v|_2^2 + |v|_4^2 + \langle A^2 v, A f(v + W_A)\rangle = 0. \tag{C.8}$$

Critical for bounding the inner product is the term $\langle A^2 v, A v^3\rangle$. Integration by parts yields

$$\begin{aligned}\langle A^2 v, A v^3\rangle &= 3\int_\Omega v^2 A v A^2 v \,\mathrm{d}x - 6\int_\Omega v|\nabla v|^2 A^2 v \,\mathrm{d}x \\ &= -3\int_\Omega v^2 |\nabla A v|^2 \,\mathrm{d}x + 6\int_\Omega v \nabla v (\nabla A v) A v \,\mathrm{d}x - 6\int_\Omega v|\nabla v|^2 A^2 v \,\mathrm{d}x \\ &\leq -\frac{3}{2}\int_\Omega v^2 |\nabla A v|^2 \,\mathrm{d}x + c\int_\Omega |\nabla v|^2 |A v|^2 \,\mathrm{d}x - 6\int_\Omega v|\nabla v|^2 A^2 v \,\mathrm{d}x \\ &\leq c|v|_2^2 \,\|\nabla v\|_\infty^2 + \frac{1}{4}|v|_4^2 + |v|_0^2 \,\|\nabla v\|_\infty^4 \leq c|v|_2^6 + \frac{1}{2}|v|_4^2.\end{aligned}$$

In (C.8), we set $x(t) = |v(t)|_2^2$. With the bound on the crucial term, this provides us with an estimate of the type

$$x'(t) \leq C\,(x(t)^p + 1)$$

for some constant $p > 0$. Hence, by a comparison principle, we find a positive time $t_0 > 0$ (depending on p) such that solutions of (C.8) persist at least up to the time t_0 and satisfy

$$\sup_{t \in [0, t_0]} |v(t)|_2 \leq C.$$

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